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PUKANSZKY'S CONDITION AND SYMPLECTIC INDUCTION

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Abstract

Pukanszky's condition is a condition used in obtaining representations from coadjoint orbits. In order to obtain more geometric insight into this condition, we relate it to symplectic induction. It turns out to be equivalent to the condition that the orbit in question is a symplectic subbundle of a modified cotangent bundle.

1. Introduction

One of the original goals of geometric quantization was to obtain a general method of constructing (irreducible) representations of Lie groups out of their coadjoint orbits. The idea was to generalize the Borel-Weil-Bott theorem for compact groups and Kirillov's results for nilpotent groups. Since then geometric quantization has led a somewhat dual life. On the one hand, in representation theory where it is called the orbit method (see [8] for a relatively recent review). On the other hand, in physics where it serves as a procedure that starts with a symplectic manifold (a classical theory) and creates a Hilbert space and a representation of the Poisson algebra as operators on it (the quantum theory).

Recent results in quantum reduction theory [5] allow us to show rigorously in some particular cases that geometric quantization intertwines the procedures of symplectic induction and unitary induction. Since the latter is one of the ingredients in the orbit method, this gives a geometrical insight into the "classical" part of the orbit method. In particular, it allows us to give a geometrical interpretation of Pukanszky's condition on a polarization which is completely different from the well-known interpretation that says that the coadjoint orbit contains an affine plane. In fact, we prove (Proposition 3.9) that Pukanszky's condition is equivalent to the statement that the coadjoint orbit in question is, in a noncanonical way, symplectomorphic to a symplectic subbundle of a modified cotangent bundle (where modified means that the canonical symplectic form on

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the cotangent bundle is modified by adding a closed 2-form on the base space). For real polarizations these results have also been obtained with completely different methods in [12]. Again for real polarizations we obtain as a corollary that Pukanszky's condition links the two dual lives of geometric quantization.

This paper is organized as follows. In §2.1 we recall briefly the basics of symplectic induction, and in §2.2 we show (heuristically) that geometric quantization intertwines symplectic induction and unitary induction. Then in §2.3 we prove that unitary induction from a one-dimensional unitary representation of a subgroup is equivalent to geometric quantization of an induced symplectic manifold. In §3 we use this induced symplectic manifold to prove the above-mentioned interpretation of Pukanszky's condition. In §4 we give an example of this interpretation that is particularly interesting for physics: the fact that the symplectomorphism of the orbit with modified cotangent bundle is not canonical translates as the fact that the position of a photon has no intrinsic meaning, i.e., depends heavily upon the observer. Finally in §5, an appendix, we collect some notation and conventions used throughout this paper.

2. Symplectic induction and induced representations

2.1. Symplectic induction. Let (M, ω) be a symplectic manifold and let H be a closed Lie subgroup of a connected Lie group G. Suppose H acts smoothly on M by symplectomorphisms and admits an equivariant momentum map $J_M: M \to \mathfrak{h}^*$, where $\mathfrak{h}$ denotes the Lie algebra of H. Symplectic induction ([9], [13], [25]) then constructs in a canonical way a symplectic manifold $(M_{\text{ind}}, \omega_{\text{ind}})$ on which G acts smoothly by symplectomorphisms with an equivariant momentum map $J_{\text{ind}}: M_{\text{ind}} \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G.

To construct M_{ind} one proceeds as follows. The group H acts on G by $h: g \mapsto R_{h^{-1}} g \equiv gh^{-1}$ and we denote by Φ_{T^*G} the canonical lift of this action to T^*G, equipped with its canonical symplectic form $d\vartheta_G$. We identify T^*G with $G \times \mathfrak{g}^*$ by identifying $\mathfrak{g}^*$ with the left-invariant 1-forms on G. In this trivialization the action of H on T^*G is given by

$$\Phi_{T^*G}(h)(g, \mu) = (gh^{-1}, \mathrm{Coad}_G(h)\mu),$$

where we have added the subscript G to stress that it concerns the coadjoint action with respect to the group G. This action admits a canonically defined equivariant momentum map $J_{T^*G}\colon T^*G \to \mathfrak{h}^*$ given by

$$J_{T^*G}(g, \mu) = -\iota_{\mathfrak{h}}^* \mu .$$

We denote by Φ_M the action of H on M, and we construct an action $\Phi_{\widetilde{M}}$ of H on $\widetilde{M} = M \times T^*G$ by $\Phi_{\widetilde{M}} = \Phi_M \times \Phi_{T^*G}$, i.e.,

$$\Phi_{\widetilde{M}}(h)(m, g, \mu) = (\Phi_M(h)(m), gh^{-1}, \mathrm{Coad}_G(h)\mu).$$

This action is symplectic for the symplectic form $\tilde{\omega} = \omega + d\vartheta_G$; it is a proper action because H is closed. Moreover, this action admits an equivariant momentum map $J_{\widetilde{M}} = J_M + J_{T^*G}$ for which $0 \in \mathfrak{h}^*$ is a regular value. The sought-for induced symplectic manifold $(M_{\mathrm{ind}}, \omega_{\mathrm{ind}})$ is the Marsden-Weinstein reduced symplectic manifold

$$M_{\mathrm{ind}} = J_{\widetilde{M}}^{-1}(0)/H .$$

To obtain the hamiltonian action of G on M_{ind} we make the following observations. The group G acts naturally on itself on the left; the canonical lift of this action to T^*G is hamiltonian and given by $g\colon (\hat{g}, \mu) \mapsto (g\hat{g}, \mu)$. We let G act trivially on M to obtain a hamiltonian action of G on $\widetilde{M}$ with the canonical equivariant momentum map $\widetilde{J}\colon \widetilde{M} \to \mathfrak{g}^*$ given by

$$\widetilde{J}(m, g, \mu) = \mathrm{Coad}_G(g)\mu . \tag{2.1}$$

This action commutes with the H-action on $\widetilde{M}$ and leaves $J_{\widetilde{M}}$ invariant; hence it induces a symplectic action of G on M_{ind}. Since $\widetilde{J}$ is invariant under the H-action, it descends as an equivariant momentum map for the G-action on M_{ind} which we denote by J_{ind}. This finishes the construction of the induced symplectic manifold. The following proposition describes the relation between $(M_{\mathrm{ind}}, \omega_{\mathrm{ind}})$, (M, ω), G, and H; it is a special case of a result of A. Weinstein [24].

Proposition 2.2. *M_{ind} is a fiber bundle over $T^*(G/H)$ with typical fiber M. Moreover, restriction of ω_{ind} to a fiber yields the original symplectic form ω on M.*

Proof. Let α be a connection on the principal H-bundle $G \to G/H$, i.e., α is a $\mathfrak{h}$-valued 1-form on G satisfying:

$$\forall X \in \mathfrak{h}\ \forall g \in G : \alpha_g(X) = X ,$$
$$\forall h \in H\ \forall Y \in T_gG : (R_{h^{-1}}^*\alpha)_g(Y) = \mathrm{Ad}_H(h)(\alpha_g(Y)),$$

where we interpret elements of $\mathfrak{h} \subset \mathfrak{g}$ as left-invariant vector fields on G. Restricting our attention to left-invariant vector fields, we can interpret α_g

as a projection $\alpha_g : \mathfrak{g} \to \mathfrak{h}$; by dualization we obtain a family of injections $\{\alpha_g^* : \mathfrak{h}^* \to \mathfrak{g}^* | g \in G\}$ satisfying $\iota_{\mathfrak{h}}^* \circ \alpha_g^* = \mathrm{id}_{\mathfrak{h}^*}$. Going to cotangent bundles, we consider the canonical projections $\mathrm{pr} : T^*G \equiv G \times \mathfrak{g}^* \to G$, $\pi : G \to G/H$, and $\overline{\mathrm{pr}} : T^*(G/H) \to G/H$. It is easy to verify that $\mathrm{pr} : J_{T^*G}^{-1}(0) = G \times \mathfrak{h}^0 \to G$ is the pull-back bundle of the bundle $T^*(G/H) \to G/H$ over the map π, where the projection $J_{T^*G}^{-1}(0) \to T^*(G/H)$ is just π^{*-1}.

We now note that $J_{\widetilde{M}}^{-1}(0) = \{(m, g, \mu) | J_M(m) = \iota_{\mathfrak{h}}^* \mu\} \cong M \times G \times \mathfrak{h}^0$, and we define a map $P : J_{\widetilde{M}}^{-1}(0) \to G \times \mathfrak{h}^0$ by

$$P(m, g, \mu) = (g, \mu - \alpha_g^*(J(m))).$$

The kernel of this map is obviously diffeomorphic to M, and the defining properties of a connection show that it is equivariant for the H-actions. Hence P induces a map $\overline{P} : J_{\widetilde{M}}^{-1}(0)/H = M_{\mathrm{ind}} \to J_{T^*G}^{-1}(0)/H \cong T^*(G/H)$ whose fiber is diffeomorphic to M. We thus obtain the following commutative diagram:

$$\text{(2.3)} \qquad \begin{array}{ccccc} J_{\widetilde{M}}^{-1}(0) & \xrightarrow{P} & G \times \mathfrak{h}^0 & \xrightarrow{\mathrm{pr}} & G \\ \Big\downarrow {\scriptstyle \mathrm{mod}\, H} & & \Big\downarrow {\scriptstyle \pi^{*-1}} & & \Big\downarrow {\scriptstyle \pi} \\ M_{\mathrm{ind}} & \xrightarrow[\overline{P}]{} & T^*(G/H) & \xrightarrow[\overline{\mathrm{pr}}]{} & G/H. \end{array}$$

This proves the first assertion of the proposition; the second is left to the reader.

2.2. Geometric quantization and induced representations. To forge the link between symplectic induction and induced representations, we make two additional assumptions. In the first place we assume that H is connected, and in the second place we assume that geometric quantization applied to the quadruple (M, ω, H, J_M) yields a unitary representation U_M of the Lie group H on the Hilbert space $\mathscr{H}_M$. Of course this requires additional data such as a polarization $\mathscr{F}_M$ on (M, ω), but we will not specify these explicitly. The aim now is to apply geometric quantization to the quadruple $(M_{\mathrm{ind}}, \omega_{\mathrm{ind}}, G, J_{\mathrm{ind}})$ in order to obtain a unitary representation of G.

We start by applying geometric quantization to the symplectic manifold $(\widetilde{M}, \tilde{\omega})$. We equip T^*G with the vertical polarization $\mathscr{F}_v$ and define on $\widetilde{M}$ the composite polarization $\widetilde{\mathscr{F}} = \mathscr{F}_M \oplus \mathscr{F}_v$. It is an elementary exercise in geometric quantization to prove that the Hilbert space $\widetilde{\mathscr{H}}$ obtained by quantization of $(\widetilde{M}, \tilde{\omega})$ with this polarization can be described as

$$\widetilde{\mathscr{H}} = \left\{ \psi : G \to \mathscr{H}_M \,\middle|\, \int_G \langle \psi(g), \psi(g) \rangle \, dg < \infty \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathscr{H}_M$, and dg denotes a nowhere vanishing volume form on G. For convenience we will now make the choice that dg is a left-invariant volume form (which is unique up to a nonzero real factor). When one then quantizes the action of G on $\widetilde{M}$, one obtains the unitary representation $\widetilde{U}$ of G on $\widetilde{\mathscr{H}}$ given by

$$(\widetilde{U}(g)\psi)(k) = \psi(g^{-1}k). \tag{2.4}$$

Note that this nice description is due to our particular choice of the volume dg on G.

The next step is to implement the Marsden-Weinstein reduction from $\widetilde{M}$ to M_{ind} by means of the group H. Although no proof is known, partial results obtained in [5], [6], [7], [9], and [21] all indicate that the following conjecture is true, a conjecture which describes the Hilbert space $\mathscr{H}_{\text{ind}}$ obtained by applying geometric quantization to the reduced symplectic manifold $M_{\text{ind}} = J_{\widetilde{M}}^{-1}(0)/H$.

Conjecture 2.5.

$$\mathscr{H}_{\text{ind}} \cong \{\psi \in \widetilde{\mathscr{H}} \,|\, \forall h \in H : U_{\widetilde{M}}(h)\psi = \text{Det}(\text{Ad}_H(h))^{-1/2} \cdot \psi\}.$$

In this conjecture $U_{\widetilde{M}}$ is the unitary representation of H on $\widetilde{\mathscr{H}}$ obtained by geometric quantization; note that the adjoint representation is with respect to the reducing group H. Of course this equivalence has to be read with caution because (i) in general one has to enlarge $\widetilde{\mathscr{H}}$ before there are elements satisfying the condition of the right-hand side and (ii) one then has to restrict to elements that are square-integrable with respect to a measure that is not specified in terms of $\widetilde{\mathscr{H}}$.

The representation $U_{\widetilde{M}}$ is readily calculated as being given by

$$(U_{\widetilde{M}}(h)\psi)(g) = \text{Det}(\text{Ad}_G(h))^{-1/2} \cdot U_M(h)\psi(gh).$$

The factor $\text{Det}(\text{Ad}_G(h))^{-1/2}$ is due to the fact that the left-invariant volume form dg on G is not invariant under the right-action of H, but transforms with $\text{Det}(\text{Ad}_G(h))$. Combining this with the conjecture, we find the following description of $\mathscr{H}_{\text{ind}}$:

$$\mathscr{H}_{\text{ind}} \cong \{\psi : G \to \mathscr{H}_M \,|\, \forall h \in H : \psi(gh^{-1}) = \gamma(h) \cdot U_M(h)\psi(g)\}, \tag{2.6}$$

where we have defined the function γ on H by

$$\gamma(h) = \sqrt{\frac{\mathrm{Det}(\mathrm{Ad}_H(h))}{\mathrm{Det}(\mathrm{Ad}_G(h))}}\,. \tag{2.7}$$

The scalar product on $\mathscr{H}_{\mathrm{ind}}$ can be described intrinsically by the following procedure (sketched). For $\psi, \chi \in \mathscr{H}_{\mathrm{ind}}$ we construct the volume form $dV = \langle\psi(g), \chi(g)\rangle\, dg$ on G. We then contract this volume form with the generators of the right-action of H on G to obtain a form α on G. Due to the defining property of $\mathscr{H}_{\mathrm{ind}}$, this form is closed and hence is the pull-back of a volume form $d\overline{V}$ on the quotient G/H. Integration of this volume form over G/H then gives the scalar product $\langle\psi, \chi\rangle_{\mathrm{ind}}$ on $\mathscr{H}_{\mathrm{ind}}$. We can find the usual description directly in terms of the functions ψ and χ if we introduce an auxiliary function ρ on G which is strictly positive and satisfies:

$$\forall h \in H : \rho(gh) = \gamma(h)^2 \cdot \rho(g)\,.$$

We denote by $d\mu$ the volume form on G/H obtained from the volume form $dV = \rho(g)\, dg$ on G by the procedure described above. With these preparations we have

$$\langle\psi, \chi\rangle_{\mathrm{ind}} = \int_{G/H} \frac{\langle\psi(g), \chi(g)\rangle}{\rho(g)}\, d\mu\,, \tag{2.8}$$

where we note that the quotient under the integral sign is a function on G/H, again due to the definition of $\mathscr{H}_{\mathrm{ind}}$. Finally we note that a different choice for the generators of the right-action of H on G changes the scalar product on $\mathscr{H}_{\mathrm{ind}}$ by a constant factor.

After the description of $\mathscr{H}_{\mathrm{ind}}$ as quantum Hilbert space of $(M_{\mathrm{ind}}, \omega_{\mathrm{ind}})$ we have to determine the representation of G on $\mathscr{H}_{\mathrm{ind}}$ associated to the hamiltonian action of G on M_{ind}. With reference to the same partial results as for Conjecture 2.5 and using that J_{ind} is obtained from the (globally) H-invariant momentum map $\widetilde{J}$ (formula (2.1)) one "deduces" that this representation is just the restriction of the representation $\widetilde{U}$ (formula (2.4)) of G on $\widetilde{\mathscr{H}}$ restricted to $\mathscr{H}_{\mathrm{ind}}$. If we compare this representation of G (formulas (2.4), (2.6), and (2.8)) with the standard description of the induced representation of G, induced from the unitary representation U_M of H on $\mathscr{H}_M$ ([8], [16], [23]), then we see that they are the same. In other words, geometric quantization intertwines the constructions "symplectic induction" and "induced representations," of course modulo the fact that Conjecture 2.5 is still open in the general case.

2.3. A particular case. We now consider the particular example of symplectic induction in which the original symplectic manifold is a single point. This example will play an important role in our interpretation of Pukanszky's condition. Although it might seem to be singular, the constructions all make sense. Since the action of H on $M = \{pt\}$ is trivial, a momentum map J_M for this action has a single value $\nu_0 \in \mathfrak{h}^*$. The condition that J_M is equivariant is equivalent to the condition that ν_0 is invariant under the coadjoint action of H on $\mathfrak{h}^*$:

$$J \text{ equivariant } \Leftrightarrow \nu_0 \ \mathrm{Coad}_H\text{-invariant}.$$

We continue with the symplectic manifold $(\widetilde{M}, \tilde{\omega}) = (\{pt\}\times T^*G, d\vartheta_G)$. The momentum map $J_{\widetilde{M}}: \widetilde{M} \to \mathfrak{h}^*$ is given by

$$J_{\widetilde{M}}(\mathrm{pt}, g, \mu) = \nu_0 + J_{T^*G}(g, \mu) = \nu_0 - \iota_{\mathfrak{h}}^* \mu. \tag{2.9}$$

It follows that the constraint set $J_{\widetilde{M}}^{-1}(0)$ is given by

$$J_{\widetilde{M}}^{-1}(0) = \{pt\} \times J_{T^*G}^{-1}(-\nu_0). \tag{2.10}$$

We thus see that if we drop the (now superfluous) reference to the point pt, we just have to reduce the canonical action of H on T^*G at ν_0, i.e., $M_{\mathrm{ind}} = J_{T^*G}^{-1}(-\nu_0)/H$.

We now invoke the Sternberg-Satzer-Marsden-Kummer reduction theorem ([20], [18], [1], [15]) to describe this reduced manifold. With α a connection on $G \to G/H$ as in the proof of Proposition 2.2, we define the 1-form $\alpha_{\nu_0} = \nu_0 \circ \alpha$ on G. Using that α is a connection and that ν_0 is invariant, it is elementary to show the existence of a closed 2-form β on G/H such that $d\alpha_{\nu_0} = \pi^*\beta$. Careful inspection of diagram (2.3) then proves the next proposition.

Proposition 2.11. *$\overline{P}: (M_{\mathrm{ind}}, \omega_{\mathrm{ind}}) \to (T^*(G/H), d\vartheta_{G/H} + \overline{\mathrm{pr}}^*\beta)$ is a symplectomorphism.*

We call a cotangent bundle T^*Q in which the canonical symplectic form $d\vartheta_Q$ is modified with the pull-back of a closed 2-form β on Q a *modified cotangent bundle* (note that $d\vartheta_Q + \overline{\mathrm{pr}}^*\beta$ is always symplectic). Thus Proposition 2.11 states that M_{ind} is symplectomorphic to the modified cotangent bundle $T^*(G/H)$. Note however that this symplectomorphism depends upon the choice of the connection α and hence is not canonical in the general case. The induced action of G on $T^*(G/H)$ can be described as the unique action that covers the canonical left-action of G on G/H and that is symplectic with respect to the symplectic form $d\vartheta_{G/H} + \overline{\mathrm{pr}}^*\beta$ (see also [3]).

Next we tackle the question of quantization and we start with $(M, \omega) = (\{\mathrm{pt}\}, 0)$. The Hilbert space $\mathscr{H}_M$ consists of sections of a complex line bundle over a point, i.e.,

$$\mathscr{H}_M = \mathbf{C}.$$

Geometric quantization of the momentum map J_M yields the infinitesimal representation $\tau\colon \mathfrak{h} \to \mathrm{End}(\mathscr{H}_M)$ given by

$$\tau(\xi) = i \cdot \nu_0(\xi) \in \mathbf{C} = \mathrm{End}(\mathbf{C}).$$

The assumption that geometric quantization of (M, ω) yields a unitary representation translates in this context as the assumption that this algebra representation τ can be integrated to a group representation $\chi\colon H \to U(1) \subset \mathrm{End}(\mathbf{C})$. In other words, we assume that $i \cdot \nu_0$ is the derivative (at the identity) of a character χ of H. By abuse of language we will say that $\nu_0 \in \mathfrak{h}^*$ is the infinitesimal form of the character χ.

It turns out that the assumptions we have made so far allow us to apply the results obtained in [7] and [5], results which tell us that in this particular case Conjecture 2.5 is true. We note in particular that the assumption that ν_0 is the infinitesimal form of a character χ is equivalent to the condition in [7] and [5] that the H-action lifts to a connection-preserving action on the prequantum bundle above $(T^*G, d\vartheta_G)$. We thus have proven the following proposition.

Proposition 2.12. *Let χ be a character of H, a closed and connected Lie subgroup of a connected Lie group G. Denote by $\nu_0 \in \mathfrak{h}^*$ its infinitesimal form and by U the unitary representation of G obtained by induction from χ. Then we have:*

(i) *χ is obtained by geometric quantization of the quadruple $(M, \omega, H, J_M) \equiv (\{\mathrm{pt}\}, 0, H, \nu_0)$, and*

(ii) *U is obtained by geometric quantization (using the vertical polarization) of the quadruple $(M_{\mathrm{ind}}, \omega_{\mathrm{ind}}, G, J_{\mathrm{ind}}) \equiv (T^*(G/H), d\vartheta_{G/H} + \overline{\mathrm{pr}}^*\beta, G, J_{\mathrm{ind}})$ which is obtained by symplectic induction from the quadruple $(\{\mathrm{pt}\}, 0, H, \nu_0)$.*

Remark 2.13. Without additional hypotheses the above proposition is true for geometric quantization using half-densities; for half-forms quantization additional conditions concerning metalinear structures are necessary [7]. If H is not connected, Proposition 2.11 remains true; Proposition 2.12 also remains true with half-density quantization, provided we add absolute values under the square root sign in (2.7).

3. Pukanszky's condition and the structure of coadjoint orbits

3.1. Polarizations and Pukanszky's condition. For the remainder of this paper we fix a connected Lie group G and an element $\mu_0 \in \mathfrak{g}^*$. To make notation less cumbersome, we denote the coadjoint action of G on $\mathfrak{g}^*$ by a simple dot, i.e., for $g \in G$ and $\mu \in \mathfrak{g}^*$ we have $g \cdot \mu \equiv \mathrm{Coad}_G(g)\mu$. We define $G_{\mu_0} \subset G$ as the isotropy subgroup of μ_0 with Lie algebra $\mathfrak{g}_{\mu_0} \subset \mathfrak{g}$, and we denote by $\mathscr{O}_{\mu_0} \equiv G \cdot \mu_0 \cong G/G_{\mu_0}$ the coadjoint orbit of μ_0 in $\mathfrak{g}^*$. We denote by $\mathfrak{g}^{\mathbf{C}}$ the complexification $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of $\mathfrak{g}$ with its canonical injection $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbf{C}}$ and complex conjugation $^{-} : \mathfrak{g}^{\mathbf{C}} \to \mathfrak{g}^{\mathbf{C}}$; the adjoint action of G is extended by linearity to $\mathfrak{g}^{\mathbf{C}}$.

Given a linear subspace $\mathfrak{a} \subset \mathfrak{g}$ containing $\mathfrak{g}_{\mu_0}$ we define the symplectic orthogonal $\mathfrak{a}^{\perp}$ by

$$\mathfrak{a}^{\perp} = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{a} : \mu_0([X, Y]) = 0\},$$

and we extend this notion in the obvious way to subspaces of $\mathfrak{g}^{\mathbf{C}}$ containing $\mathfrak{g}_{\mu_0} + i\mathfrak{g}_{\mu_0}$.

Lemma 3.1. $\mathfrak{g}^{\perp} = \mathfrak{g}_{\mu_0} \subset \mathfrak{a}^{\perp}$, $(\mathfrak{a}^{\perp})^{\perp} = \mathfrak{a}$, *and* $\dim \mathfrak{a} + \dim \mathfrak{a}^{\perp} = \dim \mathfrak{g} + \dim \mathfrak{g}_{\mu_0}$.

Definition 3.2. A *polarization* is a complex Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbf{C}}$ containing $\mathfrak{g}_{\mu_0} + i\mathfrak{g}_{\mu_0}$ and satisfying:

(i) $\mathfrak{h}$ is invariant under the Ad_G-action of G_{μ_0};

(ii) $\mathfrak{h}^{\perp} = \mathfrak{h}$; and

(iii) $\mathfrak{h} + \overline{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$.

Remark 3.3. Condition 3.2(ii) is usually given as two separate conditions: (ii-a) $\dim_{\mathbf{C}} \mathfrak{h} = \frac{1}{2}(\dim_{\mathbf{R}} \mathfrak{g} + \dim_{\mathbf{R}} \mathfrak{g}_{\mu_0})$ and (ii-b) $\mu_0([\mathfrak{h}, \mathfrak{h}]) = 0$.

Remark 3.4. One can easily show that polarizations as defined above are in 1-1 correspondence with G-invariant polarizations $\mathscr{F}$ on $\mathscr{O}_{\mu_0}$ in the sense of geometric quantization, the correspondence being given by $\mathrm{coad}_{\mathfrak{g}}(\mathfrak{h})\mu_0 = \mathscr{F}_{\mu_0} \subset (T_{\mu_0}\mathscr{O}_{\mu_0})^{\mathbf{C}}$. Condition (i) guarantees that $\mathscr{F}$ so defined is indeed well defined, (ii) translates to the fact that $\mathscr{F}$ is Lagrangian, and (iii) states that $\mathscr{F} + \overline{\mathscr{F}}$ is involutive.

Remark 3.5. In the special case $\mathfrak{h} = \overline{\mathfrak{h}}$ one says that $\mathfrak{h}$ is a real polarization; at the other extreme $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{g}^{\mathbf{C}}$, one calls $\mathfrak{h}$ a purely complex polarization.

To any polarization $\mathfrak{h}$ we can associate two (real) Lie subalgebras $\mathfrak{d} \subset \mathfrak{e}$ of $\mathfrak{g}$ by the relations $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ and $\mathfrak{e} = (\mathfrak{h} + \overline{\mathfrak{h}}) \cap \mathfrak{g}$. These Lie subalgebras will be fixed throughout the remaining part of this paper. We denote by $D_0 \subset E_0$ the connected Lie subgroups of G whose Lie algebras are $\mathfrak{d}$, resp. $\mathfrak{e}$. It follows from conditions (i) and (ii) that $\mathfrak{d}$ and $\mathfrak{e}$ are Lie subalgebras of $\mathfrak{g}$ containing $\mathfrak{g}_{\mu_0}$ that are invariant under the Ad_G-action of G_{μ_0}. We deduce that the subsets $D = D_0 \cdot G_{\mu_0} \subset E = E_0 \cdot G_{\mu_0}$ are subgroups of G.

We now collect some elementary facts about these objects (e.g., see [14] or [22]).

Lemma 3.6. (i) $\mathfrak{d}^\perp = \mathfrak{e}$.

(ii) *D_0 and D are closed Lie subgroups of G with Lie algebra $\mathfrak{d}$.*

(iii) *$\imath_\mathfrak{d}^* \mu_0 \in \mathfrak{d}^*$ is* Coad_D*-invariant.*

(iv) *$\mathfrak{e}^0$ and $\mu_0 + \mathfrak{e}^0$ are invariant under the* Coad_G*-action of D.*

(v) *If E is a Lie subgroup of G then its Lie algebra is $\mathfrak{e}$.*

Lemma 3.7 (Pukanszky's condition). *The following three conditions are equivalent:*

(i) $\mu_0 + \mathfrak{e}^0 \subset G \cdot \mu_0 \equiv \mathcal{O}_{\mu_0}$.

(ii) $D \cdot \mu_0 = \mu_0 + \mathfrak{e}^0$.

(iii) *$D \cdot \mu_0$ is closed in $\mathfrak{g}^*$.*

Proof. Since $\mu_0 + \mathfrak{e}^0$ is invariant under D, it follows that $D \cdot \mu_0 \subset \mu_0 + \mathfrak{e}^0$. From Lemma 3.1 we deduce that $\dim(D \cdot \mu_0) = \dim(\mu_0 + \mathfrak{e}^0)$ and thus we conclude that $D \cdot \mu_0$ is open in $\mu_0 + \mathfrak{e}^0$. Hence we find the implication (iii) $\Rightarrow$ (ii). Since the implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious, we only have to prove (i) $\Rightarrow$ (iii).

Therefore assume $\mu \in \mu_0 + \mathfrak{e}^0$ lies in the closure of $D \cdot \mu_0$. As above we have $D \cdot \mu \subset \mu_0 + \mathfrak{e}^0$ and because by hypothesis $\mu \in \mathcal{O}_{\mu_0}$ we still have $\dim(D \cdot \mu) = \dim(\mu_0 + \mathfrak{e}^0)$. It follows that $D \cdot \mu$ is also open in $\mu_0 + \mathfrak{e}^0$ and thus $D \cdot \mu$ intersects $D \cdot \mu_0$. It then follows immediately from the existence of $g \in G$ with $\mu = g \cdot \mu_0$ that $\mu \in D \cdot \mu_0$.

Remark 3.8. The definition of Pukanszky's condition as given above is the one used by M. Vergne [22] in the context of solvable Lie groups; there are other versions of this condition oriented more toward Lie groups that may have semisimple subgroups. One by M. Duflo [4] is that $H \cdot \mu_0$ should be closed in $(\mathfrak{g}^\mathbf{C})^*$, and one by B. Kostant [14] is that $E \cdot \mu_0$ should be closed in $\mathfrak{g}^*$. In the case of a real polarization all these conditions imply the original one of Pukanszky [17].

3.2. The structure of coadjoint orbits. We saw in the previous subsection that D is a closed subgroup of G and that $\nu_0 = \imath_\mathfrak{d}^* \mu_0 \in \mathfrak{d}^*$ is Coad_D-invariant. We thus can apply symplectic induction from a point as explained in §2.3 with D as the closed subgroup H. In this case we have $J_{\widetilde{M}}^{-1}(0) \cong J_{T^*G}^{-1}(-\nu_0) \cong G \times (\mu_0 + \mathfrak{d}^0)$, while the subset $\mu_0 + \mathfrak{e}^0 \subset \mu_0 + \mathfrak{d}^0$ is also invariant under the D-action. Hence we can enlarge and simplify the commutative diagram (2.3) to:

$$\begin{array}{ccccccc} G \times (\mu_0 + \mathfrak{e}^0) & \longrightarrow & G \times (\mu_0 + \mathfrak{d}^0) & \xrightarrow{P} & G \times \mathfrak{d}^0 & \xrightarrow{\mathrm{pr}} & G \\ \big\downarrow \scriptstyle{\mathrm{mod}\, D} & & \big\downarrow \scriptstyle{\mathrm{mod}\, D} & & \big\downarrow \scriptstyle{\pi^{*-1}} & & \big\downarrow \scriptstyle{\pi} \\ F & \longrightarrow & M_{\mathrm{ind}} & \xrightarrow[\overline{P}]{} & T^*(G/D) & \xrightarrow[\overline{\mathrm{pr}}]{} & > G/D \end{array}$$

with $F = (G \times (\mu_0 + \mathfrak{e}^0))/D$. Since we apply symplectic induction from a point, the maps P and $\overline{P}$ are diffeomorphisms. Moreover, each D-orbit intersects $\{g\} \times (\mu_0 + \mathfrak{d}^0)$ in a single point and P is affine on it; hence we can identify F as a subbundle of $T^*(G/D)$.

We now restrict the canonical symplectic form $d\vartheta_G$ of $T^*G \cong G \times \mathfrak{g}^*$ to the subspace $G \times (\mu_0 + \mathfrak{e}^0)$. We leave it to the reader to verify that the leaves of the characteristic foliation of this restricted 2-form are exactly the orbits of D_0. It then follows that F can be identified as a *symplectic* subbundle of $(T^*(G/D), d\vartheta_{G/D} + \overline{\mathrm{pr}}^*\beta)$; the induced symplectic form ω_F on F being the restriction of $d\vartheta_{G/D} + \overline{\mathrm{pr}}^*\beta$ to $F \subset T^*(G/D)$.

Since the (left) action of G on T^*G obviously preserves $G \times (\mu_0 + \mathfrak{e}^0)$, we obtain an induced symplectic action of G on F; its equivariant momentum map J_F is obtained from the momentum map $\widetilde{J}$ defined in (2.1).

Proposition 3.9. *The following four conditions on the polarization are equivalent:*

(i) *Pukanszky's condition.*
(ii) *The momentum map* $J_F : F \to \mathfrak{g}^*$ *is onto* $\mathscr{O}_{\mu_0}$.
(iii) *The symplectic action of* G *on* F *is transitive.*
(iv) J_F *is a symplectomorphism between* (F, ω_F) *and* $\mathscr{O}_{\mu_0}$.

Proof. From the definitions of F and J_F we deduce that

$$\mathrm{im}(J_F) = \{g \cdot \mu \mid g \in G,\ \mu \in \mu_0 + \mathfrak{e}^0\}.$$

In particular, $\mu_0 \in \mathrm{im}(J_F)$, and thus $\mathscr{O}_{\mu_0} \subset \mathrm{im}(J_F)$. The equivalence (i) $\Leftrightarrow$ (ii) now follows immediately from 3.7(i). Since the implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious, it suffices to show the implication (i) $\Rightarrow$ (iv).

To that end, consider $(g, \mu), (\hat{g}, \hat{\mu}) \in G \times (\mu_0 + \mathfrak{e}^0)$ that have the same image under J_F, i.e., $g \cdot \mu = \hat{g} \cdot \hat{\mu}$. From 3.7(ii) we deduce the existence of $d, \hat{d} \in D$ such that $\mu = d \cdot \mu_0$, $\hat{\mu} = \hat{d} \cdot \mu_0$, and hence we have $d^{-1} g^{-1} \hat{g} \hat{d} \in G_{\mu_0}$. Since $G_{\mu_0} \subset D$, it follows that $d_0 = g^{-1} \hat{g} \in D$ and thus $(g, \mu) = \Phi_{T^*G}(d_0)(\hat{g}, \hat{\mu})$, i.e., (g, μ) and $(\hat{g}, \hat{\mu})$ lie in the same D-orbit. Together with (i) this shows that J_F maps F bijectively to $\mathcal{O}_{\mu_0}$. It is then standard to show that it is a symplectomorphism, and thus we have shown the implication (i) $\Rightarrow$ (iv).

Remark 3.10. If $\mathfrak{h}$ is a real polarization, the two subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ are the same. In that case Pukanszky's condition states that $\mathcal{O}_{\mu_0}$ is isomorphic to the (full) modified cotangent bundle $T^*(G/D)$. At the other extreme when $\mathfrak{h}$ is purely complex, Pukanszky's condition is always satisfied and the above proposition reduces to the rather trivial statement that $\mathcal{O}_{\mu_0}$ is symplectomorphic to the zero section of the modified cotangent bundle $T^*\mathcal{O}_{\mu_0}$.

Remark 3.11. If G is an exponential group, the orbit method proceeds as follows. One assumes the polarization to be real and such that there exists a global character χ of D with $d\chi = i\iota_{\mathfrak{d}}^* \mu_0$. The representation of G associated to the orbit $\mathcal{O}_{\mu_0}$ then is the representation obtained by unitary induction from χ. Pukanszky [17] has shown that this representation is irreducible if and only if the condition that bears his name is satisfied.

However, there is another representation of G we can associate to this orbit, i.e., the one obtained by geometric quantization (using the given real polarization). Combining Propositions 2.12 and 3.9 we see that these two representations of G coincide if Pukanszky's condition is satisfied. This last result thus provides an even stronger link between geometric quantization and the orbit method.

Without additional assumptions not much more can be said about the geometric implications of Pukanszky's condition. However, if the subgroup E of G happens to be closed, we have the following proposition.

Proposition 3.12. *If E is a closed subgroup of G, and if Pukanszky's condition is satisfied, then there exists a commutative diagram*

$$\begin{array}{ccccc} T^*(G/D) & \xleftarrow{\;i\;} & \mathcal{O}_{\mu_0} & \xrightarrow{\;\overline{P}_E\;} & T^*(G/E) \\ \downarrow & & \downarrow f & & \downarrow \pi \\ G/D & = & G/D & \longrightarrow & G/E \end{array}$$

with the following properties:

(i) *(i, f) is the identification of $\mathcal{O}_{\mu_0}$ as a symplectic subbundle of $T^*(G/D)$ according to Proposition* 3.9, *and*

(ii) *$\overline{P}_E$ is a fiber bundle whose fibers, together with the restricted symplectic form, are symplectomorphic to the pseudo-Kähler space E/D.*

Proof. Define $\rho_0 = \iota_\mathfrak{e}^* \mu_0 \in \mathfrak{e}^*$, and denote by $\mathcal{O}_{\rho_0}$ the orbit of ρ_0 in $\mathfrak{e}^*$ under the Coad_E-action. We compute the isotropy subgroup of ρ_0 in E as follows. $e \in E$ lies in the isotropy subgroup of ρ_0 iff $e \cdot \mu_0 - \mu_0 \in \mathfrak{e}^0$. According to Pukanszky's condition this is equivalent to $e \cdot \mu_0 \in D \cdot \mu_0$. Since $G_{\mu_0} \subset D \subset E$ we deduce that the isotropy subgroup is D.

We now consider symplectic induction from the subgroup $E \subset G$ with $M = \mathcal{O}_{\rho_0}$. From Proposition 2.2 we deduce that M_{ind} fibers over $T^*(G/E)$ with symplectic fiber $\mathcal{O}_{\rho_0} \cong E/D$. Since one can show (e.g., [14]) that $\mathcal{O}_{\rho_0} \cong E/D$ admits a pseudo-Kähler structure, it thus only remains to show that M_{ind} is symplectomorphic to $\mathcal{O}_{\mu_0}$ and that the diagram containing $\overline{P}_E$ commutes.

To that end we investigate $J_{\widetilde{M}}^{-1}(0)$, which is given by

$$J_{\widetilde{M}}^{-1}(0) = \{(\rho, g, \mu) \in \mathfrak{e}^* \times G \times \mathfrak{g}^* | \iota_\mathfrak{e}^* \mu = \rho \in \mathcal{O}_{\rho_0}\}$$

(note that J_M is the identity map for coadjoint orbits). Now if $\rho \in \mathcal{O}_{\rho_0}$ then there exists $e \in E : \rho = \iota_\mathfrak{e}^*(e \cdot \mu_0)$ and thus we find the condition $\mu - e \cdot \mu_0 \in \mathfrak{e}^0$, or equivalently (using Pukanszky's condition) $\mu = ed \cdot \mu_0$ for some $d \in D$. Since $D \subset E$ we thus find

$$J_{\widetilde{M}}^{-1}(0) = \{(\iota_\mathfrak{e}^*(e \cdot \mu_0), g, e \cdot \mu_0) | g \in G,\ e \in E\}.$$

It then follows (with the same techniques as in the proof of 3.9) that $J_{\mathrm{ind}} \colon M_{\mathrm{ind}} \to \mathcal{O}_{\mu_0}$ is a symplectomorphism. The map $\overline{P}_E$ now is the composition of J_{ind}^{-1} with the map $\overline{P}$ from Proposition 2.2 for this induction.

Tracing diagram (2.3) for the two different symplectic inductions, one finds that the projection $f \colon F \cong \mathcal{O}_{\mu_0} \to G/D$ maps the element $g \cdot \mu_0 \in \mathcal{O}_{\mu_0}$ to $[g]_D \in G/D$ and that $\pi \circ \overline{P}_E$ maps it to $[g]_E \in G/E$, and thus the given diagram is commutative.

4. Pukanszky's condition and localization of massless particles

We tackle here, in purely geometric terms, the question of localization of massless relativistic particles in the light of our interpretation of Pukanszky's condition.

Let $\mathbf{R}^{3,1} = (\mathbf{R}^4, g)$ denote flat space-time whose metric g has the Lorentz signature $(- - - +)$. We also assume for convenience that $\mathbf{R}^{3,1}$ is oriented and time oriented as well. The group G of interest to us is the neutral component of the Poincaré group $\mathrm{Isom}(\mathbf{R}^{3,1})$, i.e., $G = O(3,1)_0 \circledS \mathbf{R}^{3,1}$. We will denote by $\xi = (\Lambda, \Gamma)$ a typical element of $\mathfrak{g} = o(3,1) \circledS \mathbf{R}^{3,1}$. Likewise, a point in $\mathfrak{g}^*$ is a pair $\mu = (M, P)$ with $M \in o(3,1)$ and $P \in \mathbf{R}^{3,1}$—interpreted as the angular and linear momentum respectively—where the pairing with $\mathfrak{g}$ is given by $\langle \mu, \xi \rangle = -\frac{1}{2}\mathrm{Tr}(M\Lambda) - g(P, \Gamma)$.

According to the point of view espoused in [19], the coadjoint orbit $\mathscr{O}_{\mu_0}$ representing the space of motions (or in other words, the classical phase space) of a massless particle with helicity $s \in \mathbf{R}\backslash\{0\}$ is specified by $\mu_0 = (M_0, P_0)$ with

$$\star(M_0)P_0 = sP_0, \quad \mathrm{Det}\, M_0 = 0, \quad \text{and} \quad P_0 \text{ future-pointing}, \tag{4.1}$$

where the star "$\star$" denotes the standard Hodge anti-involution of the Lorentz Lie algebra $o(3,1)$ identified with $\Lambda^2\mathbf{R}^{3,1} \cong \Lambda^2(\mathbf{R}^{3,1})^*$ by $(A \wedge B)V = g(B, V)A - g(A, V)B$. We note that the conditions (4.1) imply

$$g(P_0, P_0) = 0 \quad \text{and} \quad M_0 P_0 = 0. \tag{4.2}$$

The coadjoint action of G on $\mathfrak{g}^*$ is given by

$$\mathrm{Coad}_G(L, C)(M, P) = (LML^{-1} + C \wedge (LP), LP),$$

and an elementary (but tedious) computation shows that the istropy subgroup is given by $G_{\mu_0} \cong \mathrm{SO}(2) \times \mathbf{R}^3$.

As a next step, we consider the seven-dimensional (real) subalgebra

$$\mathfrak{d} = \{(\Lambda, \Gamma) \in \mathfrak{g} | \Lambda P_0 = 0\}. \tag{4.3}$$

Its main interest is that $\mathfrak{h} = \mathfrak{d}^{\mathbf{C}}$ is a real polarization. To prove this, we first recall the fact that for $\Lambda \in o(3,1)$ the condition $\Lambda P = 0$ implies that there exists $V \in \mathbf{R}^{3,1}$ such that $\Lambda = \star(V \wedge P)$. Using this fact, one can show that for $\xi, \xi' \in \mathfrak{d}$ there exists $Q \in \mathbf{R}^{3,1}$ such that $[\Lambda, \Lambda'] = P_0 \wedge Q$, and hence

$$\begin{aligned}\langle \mu_0, [\xi, \xi'] \rangle &= -\tfrac{1}{2}\mathrm{Tr}(M_0[\Lambda, \Lambda']) - g(P_0, \Lambda\Gamma' - \Lambda'\Gamma) \\ &= -g(Q, M_0P_0) \in (\Lambda P_0, \Gamma') - g(\Lambda' P_0, \Gamma) = 0\end{aligned}$$

because of (4.2) and (4.3). As a notable feature, this polarization satisfies Pukanszky's condition. To see this, note first that, since $\mathfrak{h}$ is real, we have

$\mathfrak{e} = \mathfrak{d}$, and, since G_{μ_0} is connected, we have $D = D_0$. Integrating the subalgebra $\mathfrak{d}$ yields the closed connected subgroup

$$D = \{(L, C) \in G | LP_0 = P_0\}.$$

With these ingredients we now compute

$$\begin{aligned}\mu_0 + \mathfrak{d}^0 &= \{(M, P_0) \in \mathfrak{g}^* | \star(M - M_0)P_0 = 0\} \\ &= \{(M, P_0) \in \mathfrak{g}^* | \exists C \in \mathbf{R}^{3,1} : M = M_0 + C \wedge P_0\} \subset D \cdot \mu_0,\end{aligned}$$

and thus $\mu_0 + \mathfrak{d}^0 = D \cdot \mu_0$.

We thus may apply Proposition 3.9 to conclude that $\mathscr{O}_{\mu_0}$ is symplectomorphic to the (modified) cotangent bundle of the forward light-cone of $\mathbf{R}^{3,1}$:

$$\mathscr{O}_{\mu_0} \cong T^*\mathscr{C} \quad \text{with } \mathscr{C} = G/D,$$

the projection $\pi: G \to \mathscr{C}$ being given by $P \equiv \pi((L, C)) = LP_0$. The base manifold $\mathscr{C} \cong \mathbf{R}^3 \backslash \{0\}$ is physically interpreted as the space of linear momentum and energy of the massless particle, whereas the typical fiber of the phase space $T^*\mathscr{C} \equiv \mathscr{C} \times \mathbf{R}^3$ may be identified with the configuration space where our massless particle dwells. Such an identification thus assigns to each point of the classical phase space of the massless particle a position in our three-dimensional space, i.e., it becomes "localized." However, we must emphasize that this localization procedure relies on a specific *noncanonical* choice for the connection α on the principal bundle $G \to G/D$ used to define the modified symplectic structure $d\vartheta_{G/D} + \overline{\mathrm{pr}}^* \beta$ of $T^*\mathscr{C}$. The fact that there does not exist a preferred G-invariant connection is due to the nonexistence of a reductive splitting $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{s}$ with $[\mathfrak{d}, \mathfrak{s}] \subseteq \mathfrak{s}$.

A straightforward calculation shows that to each future-pointing unit vector $I \in \mathbf{R}^{3,1}$, we can associate a connection $\alpha_I = (\widetilde{\Lambda}, \widetilde{\Gamma})$ on $G \to \mathscr{C}$ by

$$\widetilde{\Lambda} = L^{-1}\, dL - \frac{(L^{-1}\, dLP_0) \wedge (L^{-1} I)}{g(I, LP_0)},$$

$$\widetilde{\Gamma} = L^{-1}\, dC - \frac{g(I, C) \cdot L^{-1}\, dLP_0 - g(dLP_0, C) \cdot L^{-1} I}{g(I, LP_0)}.$$

These expressions all make sense because $g(I, LP_0) > 0$ for all $L \in O(3, 1)_0$.

More effort is needed to find out the modified symplectic structure on $T^*\mathscr{C}$, which turns out to be given by

$$\omega_I = d(Q\,dP) + s \cdot \overline{\mathrm{pr}}^* \left(\frac{\mathrm{Vol}(I, P)}{g(I, P)^3} \right), \tag{4.4}$$

where $P \in \mathscr{C}$ and $\overline{\mathrm{pr}}$ denotes the projection $T^*\mathscr{C} \to \mathscr{C}$. We have denoted by $\mathrm{Vol}(I, P)$ the 2-form on $\mathscr{C}$ obtained by contracting the prescribed volume form Vol of space-time with the vectors I and P. Note also that we have interchanged the traditional roles of the symbols P and Q: P denotes the coordinates in the base manifold and Q denotes the coordinates into the fiber.

By using the g-orthogonal decomposition $P = p + \|p\| \cdot I$, with $p \in I^{\perp}\backslash\{0\} \cong \mathbf{R}^3\backslash\{0\}$ and $q \in \mathbf{R}^3$ (our three-dimensional space), i.e., in a Lorentz frame adapted to the "observer" I, we get

$$\omega_I = -d(q\,dp) - s \cdot \frac{\mathrm{vol}(p)}{\|p\|^3},$$

where vol stands for the canonical volume element of $\mathbf{R}^3$. Following a completely different route, we thus recover the symplectic structure which is derived in [19] by means of another localization procedure.

We finish this discussion by noting that any connection α will provide us with an identification of $\mathscr{O}_{\mu_0}$ with $T^*\mathscr{C}$ equipped with a modified symplectic structure $\omega_\alpha = d\vartheta_{G/D} + \overline{\mathrm{pr}}^*\beta$. However, because of the invariance of $i_{\mathfrak{d}}^*\mu_0 \in \mathfrak{d}^*$, there will exist a 1-form ψ on $\mathscr{C}$ such that $\omega_\alpha = \omega_I + d(\overline{\mathrm{pr}}^*\psi)$. This implies that, modulo a redefinition of the canonical 1-form of $T^*\mathscr{C}$—a "gauge transformation" which reveals the affine structure of our three-dimensional space—the localization procedure we have spelled out in terms of Pukanszky's condition merely reduces to the choice of an otherwise arbitrary observer I in space-time.

Remark 4.5. For completeness, we recall that the massless coadjoint orbit with $s = 0$ corresponds to the choice of origin $\mu_0 = (0, P_0)$, where P_0 is null and future-pointing. In this case, the polarization is still given by (4.3) and all previous results hold except that our localization is now "canonical" since this orbit is symplectomorphic with $T^*\mathscr{C}$ endowed with its canonical symplectic structure.

Remark 4.6. It is worth mentioning that the position observables we have defined above do not Poisson-commute since for $u, v \in \mathbf{R}^3$ we have:

$$\{q \cdot u, q \cdot v\} = s \cdot \frac{\mathrm{vol}(p, u, v)}{\|p\|^3}.$$

We can easily single out the prequantizable massless orbits $\mathscr{O}_{\mu_0}$ as those satisfying $2s \in \mathbf{Z}$, and to these we can apply the geometric quantization procedure using the previously introduced real polarization, which is (of course) the vertical polarization of $(T^*\mathscr{C}, \omega_I)$. In doing so, we can easily quantize the *three* position observables (they preserve the G-invariant polarization) and end up with *noncommuting* position operators in the case of nonzero helicity. In this way we recover results already known to physicists (e.g., [2], [11]).

5. Appendix: Some notations and sign conventions

Notation 5.1. If $\mathfrak{a}$ is any (real) vector space, we denote by $\mathfrak{a}^*$ its dual space. If $\mathfrak{a}$ is a linear subspace of a vector space $\mathfrak{g}$, we denote the canonical injection by $\iota_{\mathfrak{a}}: \mathfrak{a} \to \mathfrak{g}$. Dual to the canonical injection we have the projection $\iota_{\mathfrak{a}}^*: \mathfrak{g}^* \to \mathfrak{a}^*$, and we denote by $\mathfrak{a}^0$ the annihilator of $\mathfrak{a}$ in $\mathfrak{g}^* : \mathfrak{a}^0 = \ker(\iota_{\mathfrak{a}}^*) = \{\mu \in \mathfrak{g}^* | \forall X \in \mathfrak{a} : \mu(X) = 0\}$.

Sign convention 5.2. Let Φ be a (left) action of a Lie group G on a symplectic manifold (M, ω), i.e., $\Phi: G \to \mathrm{Diff}(M)$ is a group homomorphism. For $X \in \mathfrak{g}$ ($\mathfrak{g}$ the Lie algebra of G) we define the fundamental vector field X_M on M as the vector field whose flow is $\Phi(\exp(Xt))$. The map $X \mapsto X_M$ so defined is a Lie algebra anti-homomorphism.

A momentum map (if it exists) is a map $J: M \to \mathfrak{g}^*$ satisfying $\iota(X_M)\omega + d(J^*X) = 0$ for all $X \in \mathfrak{g}$. It is called an equivariant momentum map if it is equivariant for the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$.

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References

[1] R. Abraham & J. E. Marsden, *Foundations of mechanics,* 2nd rev. ed., Benjamin-Cummings, Reading, MA, 1978.

[2] H. Bacry, *The position operator revisited*, Ann. Inst. H. Poincaré **49** (1988) 245–255.

[3] S. Benenti & W. M. Tulczyjew, *Cocycles of the coadjoint representation of a Lie group interpreted as differential forms,* Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (5) **10** (1986) 117–138.

[4] M. Duflo, *Construction de représentations unitaires d'un groupe de Lie*, Harmonic Analysis and Group Representations (A. Figà Talamanca, ed.) (Proc. cours d'été du CIME, Cortona, 1980), Liguri Editore, Napoli, 1982.

[5] C. Duval, J. Elhadad, M. J. Gotay, J. Śniatycki & G. M. Tuynman, *Quantization and bosonic BRST theory*, Ann. Phys. (N.Y.) **206** (1991) 1-26.

[6] C. Duval, J. Elhadad & G. M. Tuynman, *The BRS method and geometric quantization: some examples*, Comm. Math. Phys. **126** (1990) 535–557.

[7] M. J. Gotay, *Constraints, reduction and quantization*, J. Math. Phys. **27** (1986) 2051–2066.

[8] A. Guichardet, *Théorie de Mackey et méthode des orbites selon M. Duflo*, Exposition. Math. **3** (1985) 303–346.

[9] V. Guillemin & S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982) 515–538.

[10] ——, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

[11] A. Z. Jadczyk & B. Jancewicz, *Maximal localizability of photons*, Bull. Acad. Polon. Sci. **21** (1973) 477–483.

[12] S. A. Kamalin & A. M. Perelomov, *Construction of canonical coordinates on polarized coadjoint orbits of Lie groups*, Comm. Math. Phys. **97** (1985) 553–568.

[13] D. Kazhdan, B. Kostant, & S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978) 481–507.

[14] B. Kostant, *On certain unitary representations which arise from a quantization theory*, Group Representations in Math. and Phys. (V. Bargmann, ed.) (Proc. Battelle Seattle 1969 Rencontres), Lecture Notes in Phys., Vol. 6, Springer, Berlin, 1970, 237–253.

[15] M. Kummer, *On the construction of the reduced phase space of a Hamiltonian system with symmetry*, Indiana Univ. Math. J. **30** (1981) 281–291.

[16] G. W. Mackey, *Induced representations and quantum mechanics*, Benjamin, New York, 1968.

[17] L. Pukanszky, *On the theory of exponential groups*, Trans. Amer. Math. Soc. **126** (1967) 487–507.

[18] W. J. Satzer, Jr., *Canonical reduction of mechanical systems invariant under abelian group actions with an application to celestial mechanics*, Indiana Univ. Math. J. **26** (1977) 951–976.

[19] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1969.

[20] S. Sternberg, *On minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977) 5253–5254.

[21] G. M. Tuynman, *Reduction, quantization and non-unimodular groups*, J. Math. Phys. **31** (1990) 83–90.

[22] M. Vergne, *Polarisations*, Représentations des Groupes de Lie Résolubles (P. Bernat et al., eds.), Monographies de la Soc. Math. de France, No. 4, Dunod, Paris, 1972, 47–92.

[23] D. Vogan, *Unitary representations of reductive Lie groups*, Princeton University Press, Princeton, NJ, 1987.

[24] A. Weinstein, *A universal phase space for particles in Yang-Mills fields*, Lett. Math. Phys. **2** (1978) 417–420.

[25] S. Zakrzewski, *Induced representations and induced hamiltonian actions*, J. Geom. Phys. **3** (1986) 211–219.

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ON THE LAPLACIAN AND THE GEOMETRY OF HYPERBOLIC 3-MANIFOLDS

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Abstract

Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume hyperbolic 3-manifold which is homeomorphic to the interior of a compact manifold. Let $\lambda_0(N) = \inf \operatorname{spec}(-\Delta)$ where Δ is the Laplacian acting on functions on N. We prove that if N is not geometrically finite, then $\lambda_0(N) = 0$, and if N is geometrically finite we produce an upper bound for $\lambda_0(N)$ in terms of the volume of the convex core. As a consequence we see that $\lambda_0(N) = 0$ if and only if N is not geometrically finite. We also show that if N has a lower bound for its injectivity radius and is not geometrically finite, then its limit set L_Γ has Hausdorff dimension 2.

1. Introduction

In this paper we will study the relationship between the geometry of infinite volume hyperbolic 3-manifolds and the bottom λ_0 of the spectrum of the Laplacian. We will also study the relationship between spectral information and the measure-theoretic properties of the limit set. These relationships have been studied extensively by Patterson (cf. [28], [27]) and Sullivan (cf. [32], [33]), and much of this paper may be regarded as an extension of their work. Recall that a hyperbolic 3-manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. Our first result is:

Theorem A. *Let N be an infinite volume, topologically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if N is not geometrically finite. Moreover, there exists a constant K such that if N is geometrically finite, then*

$$\lambda_0(N) \le K\frac{|\chi(\partial C(N))|}{\operatorname{vol}(C(N))},$$

where $\operatorname{vol}(C(N))$ denotes the volume of N's convex core.

Combining Theorem A with work of Lax and Phillips ([20], [21]) we show that λ_0 detects whether or not a topologically tame hyperbolic 3-manifold is geometrically finite.

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Corollary B. *Let N be an infinite volume topologically tame hyperbolic 3-manifold. N is geometrically finite if and only if $\lambda_0(N) \neq 0$.*

In a forthcoming paper [5], Marc Burger and the author prove that there exists a constant $G > 0$ such that if N is geometrically finite, then

$$\lambda_0(N) \geq \frac{G}{\mathrm{vol}(\mathcal{N}_1(C(N)))^2},$$

where $\mathrm{vol}(\mathcal{N}_1(C(N)))$ denotes the volume of the neighborhood of radius one of the convex core. (This result is analogous to results of Schoen [29] and Dodziuk-Randol [13] for the closed and finite volume cases.) Thus combining this result with Theorem A we see that the volume of the convex core "controls" λ_0. Here is one intuitive explanation for such a relationship. There always exists a positive harmonic function f such that $\Delta f = -\lambda_0 f \cdot \nabla(-\log f)$ is a bounded vector field whose associated flow is volume-increasing and the rate of increase at each point is at least λ_0. Outside of the convex core the geometry is exponentially expanding, so it is easy to construct volume-increasing flows with "large" rates of volume increase. However, the convex core is more congested, and the thicker the convex core is the more difficult it will be to construct flows with a "large" rate of volume increase.

In [32], Sullivan proved that if $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-manifold, and D is the Hausdorff dimension of the limit set of Γ, then $\lambda_0(N) = 1$ if $D \leq 1$, while otherwise $\lambda_0(N) = D(2-D)$. Thus combining Theorem A with the above-mentioned result of Burger and Canary makes explicit the intuitive relationship between the thickness (volume) of the convex core and the fuzziness (Hausdorff dimension) of the limit set. (The basis of this second intuitive link is that the convex core is the quotient of the convex hull of the limit set by the group action. It stands to reason that the limit set should be locally complicated if and only if the convex core is thick.) Thus the bottom of the spectrum of the Laplacian, the Hausdorff dimension of the limit set, and the volume of the convex core all serve as measures of "how geometrically finite" N is.

By analogy one would conjecture that the limit set of a topologically tame hyperbolic 3-manifold which is not geometrically finite had Hausdorff dimension 2. We remark that it is shown in [8] that the limit set of a topologically tame hyperbolic 3-manifold either has measure zero or is the entire sphere at infinity. In this paper we prove the following result.

Theorem C. *Let $N = \mathbf{H}^3/\Gamma$ be a topologically tame hyperbolic 3-manifold with a lower bound on its injectivity radius. If N is not geomet-*

rically finite, then the limit set L_Γ *for* Γ*'s action on the sphere at infinity has Hausdorff dimension* 2.

Theorem A and Corollary B hold for analytically tame hyperbolic 3-manifolds, and Theorem C may be extended to analytically tame hyperbolic 3-manifolds with thin parts of uniformly bounded type (see §2 for definitions). The results will be stated and proved in this generality in the text.

In §2 we will review the structure of hyperbolic 3-manifolds. In §3 we will reprove an upper bound on λ_0 due to Buser and also derive a preliminary result about the growth of harmonic functions on analytically tame hyperbolic 3-manifolds. In §4, we will use Buser's upper bound to derive Theorem A and Corollary B and some other consequences of Theorem A concerning the critical exponent of the Poincaré series and the bottom of the essential spectrum. In §5 we will prove Theorem C, by showing that the harmonic function given by Patterson-Sullivan measure has subexponential growth.

2. The structure of hyperbolic 3-manifolds

Let N be a (orientable) hyperbolic 3-manifold with finitely generated fundamental group. N may be represented as the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a group Γ of orientation-preserving isometries of $\mathbf{H}^3$. We recall that the *limit set* L_Γ of Γ is defined to be the smallest closed Γ-invariant subset of the sphere at infinity S^2_∞ for hyperbolic 3-space. We will say that N is *elementary* if Γ is abelian. If N is nonelementary, the *convex core* $C(N)$ of N is defined to be the smallest convex submanifold such that the inclusion map is a homotopy equivalence. Explicitly, $C(N) = CH(L_\Gamma)/\Gamma$, where $CH(L_\Gamma)$ denotes the convex hull (in $\mathbf{H}^3$) of the limit set L_Γ. (See Maskit [24] for basic definitions in the theory of Kleinian groups.) The following structural theorem is central to understanding hyperbolic 3-manifolds and Kleinian groups.

Theorem 2.1 (Ahlfors' finiteness theorem [1]). *Let* N *be a nonelementary hyperbolic* 3*-manifold with finitely generated fundamental group. The boundary* $\partial C(N)$ *of the convex core* $C(N)$ *is a finite area hyperbolic surface, i.e., there exists a* C^0*-isometric embedding of a finite area hyperbolic surface into* N *with image* $\partial C(N)$.

This statement combines Ahlfors' finiteness theorem [1] with Thurston's observation that the boundary of the convex core is a hyperbolic surface (see Epstein-Marden [17] or Thurston [34]). Thus, in particular, $\partial C(N)$

has area $2\pi|\chi(\partial C(N))|$. If $C(N)$ has finite volume or N is elementary, then N is said to be *geometrically finite.* (If N is elementary we will say that $\mathrm{vol}(C(N)) = 0$.) Recall that N is topologically tame if it is geometrically finite (see, e.g., Marden [23]). We will say that N is *convex cocompact* if $C(N)$ is compact (i.e., if N is geometrically finite and has no cusps).

We will say that a hyperbolic 3-manifold N with finitely generated fundamental group is *analytically tame* if $C(N)$ may be exhausted by a sequence of compact submanifolds $\{C_i\}$ such that $C_i \subset C_j$ if $i < j$, $\cup C_i^0 = C(N)$ (where C_i^0 is the interior of C_i considered as a subset of $C(N)$), and there exist K and L such that the boundary ∂C_i of C_i has area at most K and the neighborhood of radius one of ∂C_i has volume at most L for all i. We only require that our submanifolds C_i have Lipschitz boundary. This regularity assumption is natural, as the boundary of the convex core itself is always a Lipschitz submanifold but is not in general a C^1-submanifold (see Epstein-Marden [17]).

In [8] the following theorem is proved.

Theorem 2.2 [8]. *If N is a topologically tame hyperbolic 3-manifold, then N is analytically tame.*

In the same paper [8] the following generalization of a result of Thurston [34] is established.

Theorem 2.3. *If N is analytically tame hyperbolic 3-manifold, then either $L_\Gamma = S^2_\infty$ or L_Γ has measure zero. Moreover, if $L_\Gamma = S^2_\infty$, then Γ acts ergodically on S^2_∞.*

Work of Bonahon guarantees that there is a large class of hyperbolic 3-manifolds which are topologically tame. (This ordering is historically misleading—Theorem 2.4 was actually used to prove Theorem 2.2; see the remarks at the end of the section for a further discussion.) Let Γ be a discrete subgroup of the group of isometries of hyperbolic 3-space. A finitely generated group Γ of hyperbolic isometries is said to satisfy condition (B) if it is not cyclic, and whenever $\Gamma = G * H$ is a nontrivial free decomposition of Γ there exists a parabolic element γ which is not conjugate to any element of G or H. In particular, condition (B) is satisfied if Γ is freely indecomposable.

Theorem 2.4 (Bonahon [3]). *If $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold and Γ satisfies condition* (B), *then N is topologically tame.*

It will be necessary in the proof of Theorem C to make use of the thick-thin decomposition of a hyperbolic 3-manifold. We recall that the injectivity radius of N at a point x, denoted $\mathrm{inj}(x)$, is defined to be half the length of the shortest (homotopically nontrivial) loop through x. There exists a constant $\mathcal{M}$, called the Margulis constant, such that if $\varepsilon < \mathcal{M}$ and

$$N_{\text{thin}(\varepsilon)} = \{x \in N \mid \text{inj}(x) \leq \varepsilon\},$$

then every component of $N_{\text{thin}(\varepsilon)}$ is either

(a) a torus cusp, i.e., a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z} \oplus \mathbf{Z}$,

(b) a rank one cusp,i.e., a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z}$, or

(c) a solid torus neighborhood of a geodesic

(see Thurston [34] or Morgan [25]). We also define

$$N_{\text{thick}(\varepsilon)} = \{x \in N \mid \text{inj}(x) \geq \varepsilon\}.$$

We further remark that if ε is chosen to be less than the Margulis constant, that there exists an $L > 0$ (depending only on ε) such that if σ is any geodesic in N, then the distance (in σ) between any two components of $\sigma \cap N_{\text{thin}(\varepsilon)}$ is at least L. (When reading about hyperbolic 3-manifolds it is often easier, on a first reading, to assume that there are no parabolics or even that there is a uniform lower bound on injectivity radius. This caution applies equally well to this paper, especially the proof of Theorem C.)

We furthermore say that N has *thin parts of uniformly bounded type* if there exists J such that if S is any component of $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$, then S has diameter less than J. In particular, if N contains any rank-one cusps, then their intersections with the convex core have finite volume; such rank-one cusps are said to be bounded or doubly cusped in the language of Kleinian groups.

Remarks. (1) Actually Bonahon [3] proved that hyperbolic 3-manifolds satisfying condition (B) are geometrically tame. The main theorem of [8] uses this theorem to prove that hyperbolic 3-manifolds are topologically tame if and only if they are geometrically tame. Analytic tameness is a consequence of geometric tameness. We have developed the structure in this way to avoid introducing simplicial hyperbolic surfaces and the more technical points in the definition of geometric tameness, none of which is necessary for the work in this paper. We urge the reader to consult Bonahon [3], Thurston [34], or Canary [8] for a discussion of geometric tameness.

(2) It is conjectured that all hyperbolic 3-manifolds with finitely generated fundamental groups are topologically tame, and hence both geometrically and analytically tame. However, there are hyperbolic 3-manifolds

which are known to be analytically tame but which are not known to be topologically tame. In particular, Culler and Shalen [12] proved that there is a dense G_δ of analytically tame manifolds in the boundary of the Schottky space of genus 2.

(3) Condition (B) is really a topological condition. Let N_ε^0 be obtained from N by removing the noncompact components of $N_{\text{thin}(\varepsilon)}$. There exists a compact submanifold C of N_ε^0 such that the inclusion map is a homotopy equivalence and C intersects each component of the boundary in either an annulus or a torus (see Feighn-McCullough [18]). Γ satisfies condition (B) if every compressible curve on the boundary ∂C of C intersects the boundary of a noncompact component of $N_{\text{thin}(\varepsilon)}$. (A curve in ∂C is said to be compressible if it is homotopically trivial in C, but not in ∂C.)

3. Buser's upper bound for λ_0

Let N be a complete Riemannian n-manifold (without boundary). We recall some equivalent definitions of $\lambda_0(N)$ (in this paper the Laplacian $\Delta f = \operatorname{div}(\operatorname{grad} f)$ is a negative definite operator):

$$\begin{aligned}\lambda_0(N) &= \sup\{\lambda \mid \exists f \in C^\infty(N) \text{ s.t. } \Delta f = -\lambda f \text{ and } f > 0\} \\ &= \inf_{f \in C_0^\infty(N)} \left(\frac{\int_N |\nabla f|^2}{\int_N f^2} \right) \\ &= \inf \operatorname{spec}(-\Delta).\end{aligned}$$

We also recall that the Cheeger constant $h(N)$ is defined to be the infimum, over all compact n-submanifolds A of N (with Lipschitz boundary), of $\operatorname{vol}_{n-1}(\partial A)/\operatorname{vol}(A)$. Buser [6] proved that if N has Ricci curvature bounded from below, then $h(N)$ gives an upper bound for $\lambda_0(N)$. (In Cheeger's original paper [9] he proved that the Cheeger constant gives a lower bound on λ_0 with no constraints on the geometry of the manifold, to be precise $\lambda_0(N) \geq h(N)^2/4$.) We give a new proof of this upper bound which also yields a L^2-bound on the growth rate of harmonic functions on analytically tame hyperbolic 3-manifolds.

Theorem 3.1 (Buser [6]). *If the Ricci curvature of a complete Riemannian n-manifold N (without boundary) is bounded below by $-(n-1)\kappa^2$, then*

$$\lambda_0(N) \leq R\kappa h(N),$$

where R depends only on n.

Proof of 3.1. We may assume, by scaling the metric, that $\kappa = 1$. Then Cheng's comparison principle [10] assures us that $\lambda_0(N) \leq (n-1)^2/4$. Let f be a positive eigenfunction of the Laplacian with eigenvalue $-\lambda_0$ (see either Cheng-Yau [11] or Sullivan [38] for a proof that f exists). Now the infinitesimal Harnack inequality of Yau [36] implies that $|\frac{\nabla f}{f}(x)| \leq R$ for some R depending only on n and all $x \in N$.

Now consider $\log f \cdot \nabla \log f = \nabla f / f$ and

$$\Delta \log f = -\lambda_0 - \frac{|\nabla f|^2}{f^2} \leq -\lambda_0.$$

Let A be a compact n-submanifold of N. Then by Stokes' theorem,

$$\int_A \Delta(-\log f) = \int_{\partial A} -\frac{\nabla f}{f} \cdot \hat{n}.$$

But

$$\int_A \Delta(-\log f) \geq \lambda_0 \operatorname{vol}(A)$$

and

$$\int_{\partial A} -\frac{\nabla f}{f} \cdot \hat{n} \leq R \operatorname{vol}_{n-1}(\partial A),$$

so

$$R \frac{\operatorname{vol}_{n-1}(\partial A)}{\operatorname{vol}(A)} \geq \lambda_0,$$

which completes the proof. q.e.d.

When N is analytically tame and h is a positive harmonic function, the same argument applies to prove:

Proposition 3.2. *If N is an analytically tame hyperbolic 3-manifold and h is a positive harmonic function on N, then*

$$\int_{C(N)} \left| \frac{\nabla h}{h} \right|^2 < \infty.$$

Proof of 3.2. Let C_i be a sequence of compact submanifolds exhausting $C(N)$ such that ∂C_i has area less than K. Then

$$\int_{C_i} \Delta(-\log h) = \int_{C_i} \left| \frac{\nabla h}{h} \right|^2 = \int_{\partial C_i} -\frac{\nabla h}{h} \cdot \hat{n} \leq \operatorname{area}(\partial(C_i))R.$$

Therefore,

$$\int_{C(N)} \left| \frac{\nabla h}{h} \right|^2 \leq KR < \infty.$$

Remark. It is a consequence of Theorem 1.2 in Li-Yau [22] that if $u(x, t)$ is any positive solution of the heat equation $(\Delta - \frac{\partial}{\partial t})u(x, t) = 0$ on

$N \times (0, \infty)$, where N is a complete noncompact Riemannian n-manifold without boundary whose Ricci curvature is bounded below by $-(n-1)$, then

$$\frac{|\nabla u|^2}{u^2} - \frac{\alpha u_t}{u} \le \frac{n\alpha^2(n-1)}{\sqrt{2}(\alpha-1)} + \frac{n\alpha^2}{2t}$$

for all $\alpha > 1$. If f is a positive eigenfunction of the Laplacian with eigenvalue $-\lambda$ on N, then $u(x,t) = e^{-\lambda t} f(x)$ is a positive solution of the heat equation. Applying the above inequality to u and letting $\alpha = 2$ and t go to ∞, we obtain

$$\frac{|\nabla f|^2}{f^2} \le 2\sqrt{2}n(n-1) - 2\lambda.$$

Therefore in our proof of Buser's theorem we may take R to be $\sqrt{2\sqrt{2}n(n-1)}$. This appears to improve on the constant obtained in Buser's original paper [6].

4. Proofs of Theorem A and Corollary B

Theorem A. *Let N be an infinite volume, analytically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if N is not geometrically finite. Moreover, there exists $K > 0$ such that if N is geometrically finite, then*

$$\lambda_0(N) \le K \frac{|\chi(\partial C(N))|}{\text{vol}(C(N))}.$$

Proof of Theorem A. We first suppose that $C(N)$ has infinite volume (i.e., that N is not geometrically finite). Let $\{C_i\}$ be a collection of compact submanifolds exhausting $C(N)$ such that $\text{area}(\partial C_i) \le K$ for some K. In this case, $\lim_{i \mapsto \infty} \text{vol}(C_i) = \infty$, so $h(N) = 0$. Therefore, applying Buser's Theorem 3.1, we see that $\lambda_0(N) = 0$.

If N is geometrically finite, let $C_\varepsilon = C(N) \cap N_{\text{thick}(\varepsilon)}$. Since $C(N) \cap N_{\text{thick}(\varepsilon)}$ is compact, $N_{\text{thin}(\varepsilon)}$ has only finitely many components, so there exists $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$ then all the components of $N_{\text{thin}(\varepsilon)}$ are noncompact. Let T be a noncompact component of $N_{\text{thin}(\varepsilon)}$. If T is a torus cusp, then it is isometric to $T^2 \times [c, \infty)$ with the metric

$$ds^2 = \frac{ds_{T^2}^2 + dt^2}{t^2},$$

where $ds_{T^2}^2$ is a Euclidean metric on T^2 and $c > 0$. If T is a rank-one cusp, then $T \cap C(N)$ is isometric to $A \times [c, \infty)$ with metric

$$ds^2 = \frac{ds_A^2 + dt^2}{t^2},$$

where ds_A^2 is an Euclidean metric on the annulus A and $c > 0$. If $\varepsilon \le \varepsilon_0$, then $\partial C_\varepsilon - \partial C(N) = \partial N_{\text{thin}(\varepsilon)} \cap C(N)$, so

$$\text{area}(\partial C_\varepsilon - \partial C(N)) = \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 \text{area}(\partial C_{\varepsilon_0} - \partial C(N)),$$

which implies

$$\lim_{\varepsilon \mapsto 0} \text{area}(\partial C_\varepsilon) = \text{area}(\partial C(N)) = 2\pi|\chi(\partial C(N))|,$$

while

$$\lim_{\varepsilon \mapsto 0} \text{vol}(C_\varepsilon) = \text{vol}(C(N)).$$

Therefore,

$$h(N) \le \frac{2\pi|\chi(\partial C(N))|}{\text{vol}(C(N))}$$

and one may again use Theorem 3.1 to complete the argument. q.e.d.

In a series of papers Lax and Phillips ([20], [21]) have studied the spectrum of the Laplacian on finite volume geometrically finite hyperbolic manifolds. In particular, they proved that $\lambda_0 \neq 0$. (One may also see that $\lambda_0 \neq 0$ for geometrically finite hyperbolic 3-manifolds using the techniques of Patterson and Sullivan; see, for example, [32], [33].) We state a portion of their results.

Theorem 4.1 (Lax and Phillips). *Let N be an infinite volume geometrically finite hyperbolic 3-manifold. The intersection of* spec$(-\Delta)$ *with the interval* $[0, 1)$ *consists entirely of a finite number of point eigenvalues* (*of finite multiplicity*) *all lying in* $(0, 1)$, *and there are no point eigenvalues in* $[1, \infty)$. *Moreover, the spectrum is absolutely continuous and of infinite uniform multiplicity in* $[1, \infty)$.

We combine this with Theorem A to obtain:

Corollary B. *Let N be an infinite volume analytically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if and only if N is not geometrically finite.*

We recall that the critical exponent of the Poincaré series of a Kleinian group Γ is defined to be

$$\delta = \inf\left\{ s \,\middle|\, \sum_{\gamma \in \Gamma} e^{-sd(0, \gamma(0))} < \infty \right\}.$$

This critical exponent is closely related to λ_0. In fact (see Sullivan [33]), if $\lambda_0 = 1$ then $\delta \leq 1$, otherwise $\delta > 1$ and $\lambda_0 = \delta(2-\delta)$. Therefore Theorem A implies:

Corollary 4.2. *If $N = \mathbf{H}^3/\Gamma$ is analytically tame but not geometrically finite, then the critical exponent of its Poincaré series is 2. Moreover, if N is geometrically finite and $\lambda_0(N) \neq 1$, then*

$$\delta \geq 2 - \frac{K|\chi(\partial C(N))|}{\operatorname{vol} C(N)}.$$

Let n_k denote the number of elements γ of Γ such that $\gamma(0)$ is contained in the ball of (hyperbolic) radius k about 0. Then

$$\delta = \limsup \frac{\log n_k}{k}.$$

If N is further convex cocompact, then there exist constants a and A such that $ae^{k\delta} \leq n_k \leq Ae^{k\delta}$ (see Sullivan [30]). So we can interpret Corollary 4.2 as an asymptotic estimate on the number of lattice points in the ball of radius k in terms of the volume of the convex core.

We recall that the discrete spectrum of N is defined to be the isolated points in spec$(-\Delta)$ which correspond to eigenvalues of finite multiplicity. The *essential spectrum* of N is the complement of the discrete spectrum in spec$(-\Delta)$. (See Donnelly [14] for a discussion of the essential spectrum.) One direct consequence of Lax and Phillips' result is that the bottom $\lambda_0^{\mathrm{ess}}(N)$ of the essential spectrum is 1, whenever N is geometrically finite. As a consequence of Theorem A we obtain:

Corollary 4.3. *If N is analytically tame, but not geometrically finite, then $\lambda_0^{\mathrm{ess}}(N) = 0$.*

Proof of 4.3. Theorem A assures us that $0 \in \operatorname{spec}(-\Delta)$. But it is a result of Yau [37] that there are no (nonzero) harmonic functions in $L^2(N)$ when N is a complete infinite volume Riemannian manifold. Therefore 0 is in the essential spectrum. q.e.d.

Another consequence of Theorem A and Lax and Phillips' result is:

Corollary 4.4. *If N is geometrically finite and*

$$\operatorname{vol}(C(N)) > \frac{1}{2\pi K|\chi(\partial C(N))|},$$

then N has nonempty point spectrum.

Remarks. (1) If N is a geometrically finite hyperbolic 3-manifold, then there exists a positive eigenfunction f such that $\Delta f = -\lambda_0 f$ and

$$f(x) = \int_{S^2_\infty} \left(\frac{1-|x|^2}{|x-\xi|^2}\right)^d d\nu,$$

where $1 \leq d < 2$, and ν is a probability measure on the sphere at infinity for the Poincaré ball model for $\mathbf{H}^3$ (see Sullivan [30], Patterson [28], or Nicholls [26]). In particular, $|\nabla f/f| \leq 2$. Therefore, returning to the proof of Buser's theorem, we see that $\lambda_0(N) \leq 2h(N)$, so that the constant K in Theorem A may be taken to be 4π. Notice that the topological term $|\chi(\partial C(N))|$ is necessary in the statement of Theorem A, since when one passes to a finite cover of a geometrically finite hyperbolic 3-manifold λ_0 remains the same.

(2) In [31] Sullivan proves that $\lambda_0(N) = 0$ when N is a "hyperbolic half-cylinder" (see remark (1) at the end of §5 for the definition of a hyperbolic half-cylinder and a discussion of Sullivan's work). Let M be a hyperbolic 3-manifold which fibers over the circle. C. L. Epstein [16] proved that $\lambda_0(N) = \lambda_0^{\mathrm{ess}}(N) = 0$ if N is the cover of M associated to the fiber subgroup. In both cases the manifolds involved are known to be topologically tame.

(3) In [15] Doyle shows that there exists $Y > 0$ such that if Γ is a classical Schottky group, then $\lambda_0(N) \geq Y$, where $N = \mathbf{H}^3/\Gamma$. Therefore, Theorem A implies that $\mathrm{vol}(C(N)) \leq K(2g-2)/Y$, where g is the genus of the classical Schottky group. We can interpret this as a quantitative version of the fact (see Jorgensen-Marden-Maskit [19]) that all algebraic limits of classical Schottky groups are geometrically finite. Recall that Γ is a classical Schottky group of genus g if there exist g mutually disjoint pairs of circles in the sphere at infinity such that Γ is generated by a set of g Möbius transformations each of which takes the interior of a circle to the exterior of its partner circle. In this case, the neighborhood of radius 1 of $C(N)$ is a handlebody of genus g.

(4) If N is convex cocompact and $\lambda_0(N) = 1$, then Γ is either a Schottky group (i.e., the neighborhood of radius 1 of $C(N)$ is a handlebody) or a Fuchsian group (i.e., $C(N)$ is a totally geodesic surface) (see Sullivan [30] or Braam [4]). Presumably, the case with cusps is equally restrictive.

5. The Hausdorff dimension of the limit set

In this section we prove that the limit sets of geometrically infinite, analytically tame hyperbolic 3-manifolds with thin parts of uniformly bounded type have Hausdorff dimension 2. If the limit set is not all of S^2, then such limit sets provide naturally arising examples of sets with measure zero but Hausdorff dimension 2. This phenomenon was first studied by Sullivan [31], who proved this result for "hyperbolic half-cylinders" (see the remarks at the end of the section for a discussion of Sullivan's work).

Theorem C. *If $N = \mathbf{H}^3/\Gamma$ is an analytically tame hyperbolic 3-manifold with thin parts of uniformly bounded type which is not geometrically finite, then its limit set L_Γ has Hausdorff dimension 2.*

Proof of Theorem C. The first step in the Patterson-Sullivan program is the construction of a probability measure μ on S^2_∞ (called Patterson-Sullivan measure) supported on the limit set such that

$$(*) \qquad \mu(\gamma(E)) = \int_E |\gamma'|^\delta \, d\mu,$$

where δ is the critical exponent of the Poincaré series, E is any Borel subset of the sphere, and γ is any element of Γ. (All calculations are done in the Poincaré ball model for hyperbolic 3-space.) We also recall that if ν is any probability measure on S^2_∞ and $d \in [0, 2]$, then we may define a function $\phi_{\nu, d}$ on $\mathbf{H}^3$, where

$$\phi_{\nu, d}(x) = \int_{S^2_\infty} |\alpha'_x|^d \, d\nu,$$

α_x being a hyperbolic isometry taking x to 0. Explicitly,

$$|\alpha'_x(\xi)| = \frac{1 - |x|^2}{|x - \xi|^2}.$$

$\phi_{\nu, d}$ is then a positive eigenfunction of the Laplacian with eigenvalue $d(d-2)$. When μ is the Patterson-Sullivan measure on L_Γ and $d = \delta$, condition $(*)$ guarantees that $\phi_{\mu, \delta}$ is equivariant with respect to Γ. Recall from Corollary 4.2 that in our case $\delta = 2$, so $\phi_{\mu, 2}$ descends to a positive harmonic function on N. See Sullivan [30], [32], Patterson [28], or Nicholls [26] for a discussion of Patterson-Sullivan measure.

Our proof depends on the following result of Sullivan (see Theorem 2.15 of [33] or [31]) whose proof we will review. Recall that if ϕ is a function on $\mathbf{H}^3$, we define its exponential growth rate to be

$$e(\phi) = \limsup_{R \to \infty} \left(\frac{\log(\max\{\phi(x) \mid x \in B_R(0)\})}{R} \right).$$

If $e(\phi) \leq 0$, then ϕ is said to have *subexponential growth.*

Proposition 5.1 (Sullivan). *Let ν be a probability measure on S^2_∞, and $\phi(x) = \int_{S^2_\infty} |\alpha'_x|^2 \, d\nu$. If ϕ has subexponential growth, then the support of ν has Hausdorff dimension* 2.

Proof of 5.1. If $\xi \in S^2_\infty$, let $\nu(\xi, r)$ denote the ν-mass of a disk $B(\xi, r)$ of (spherical) radius r about ξ. Given $\varepsilon > 0$ choose $T(\varepsilon)$ such that if $d(0, x) \geq T(\varepsilon)$ then $\phi(x) \leq e^{\varepsilon d(0,x)}$.

Lemma 5.2. *There exists* $C > 0$ *such that if* $r \leq e^{-T(\varepsilon)}$ *and* $\xi \in S^2_\infty$, *then* $\nu(\xi, r) \leq Cr^{2-\varepsilon}$.

Proof of 5.2. Let $\sigma(\xi)$ denote the geodesic ray from 0 to ξ, and let $p(\xi, r)$ denote the point along this ray at a distance of $-\log(r)$ away from the origin. Then $p(\xi, r) = ((1-r)/(1+r))\xi$, so

$$|\xi - p(\xi, r)| = \frac{2r}{1+r} \leq 2r.$$

If $\hat{\xi} \in B(\xi, r)$, then $|\hat{\xi} - p(\xi, r)| \leq 3r$. Moreover

$$1 - |p(\xi, r)|^2 = \frac{4r}{(1+r)^2}.$$

If $\hat{\xi} \in B(\xi, r)$ and α_p is a hyperbolic isometry taking $p(\xi, r)$ to 0, then

$$|\alpha_p'(\hat{\xi})| \geq \frac{4r}{9r^2(1+r)^2} \geq \frac{1}{9r}$$

since $r \leq 1$. Therefore,

$$\phi(p(\xi, r)) \geq \int_{B(\xi, r)} |\alpha_p'| \, d\nu \geq \frac{1}{81r^2}\nu(\xi, r),$$

which implies

$$\nu(\xi, r) \leq 81r^2\phi(p(\xi, r)) \leq 81r^2e^{-\varepsilon \log r} = 81r^{2-\varepsilon}. \quad \text{q.e.d.}$$

If $\{B(\xi, r_i)\}$ is a covering of the support of ν by a countable collection of balls of radius $r_i \leq e^{-T(\varepsilon)}$ centered at ξ_i, then

$$1 \leq \sum \nu(\xi, r_i) \leq \sum Cr_i^{2-\varepsilon},$$

which shows that $\operatorname{supp}(\nu)$ has positive $(2-\varepsilon)$-dimensional Hausdorff measure. In particular, the Hausdorff dimension of $\operatorname{supp}(\nu)$ is at least $2-\varepsilon$. But since this holds for all $\varepsilon > 0$, $\operatorname{supp}(\nu)$ has Hausdorff dimension 2, and hence Proposition 5.1 is proved.

The proof of Theorem C is then completed by the following proposition.

Proposition 5.3. *Let* N *be an analytically tame hyperbolic* 3*-manifold with thin parts of uniformly bounded type which is not geometrically finite, and* μ *its associated Patterson-Sullivan measure. Then* $\phi_{\mu,2}$ *has subexponential growth.*

Proof of 5.3. Throughout this proof we will fix a value of $\varepsilon > 0$ which is less than the Margulis constant. For convenience we will assume that $0 \in C(N) \cap N_{\text{thick}(\varepsilon)}$, and ϕ will serve as shorthand for $\phi_{\mu,2}$.

We will need the following easy consequence of elliptic theory:

Lemma 5.4. *Given any $\delta > 0$ and $\varepsilon > 0$, there exists $A > 0$ with the following property. If M is any complete hyperbolic 3-manifold, $x \in M_{\text{thick}(\varepsilon)}$, and h is any positive harmonic function on M such that $|\frac{\nabla h}{h}(x)| \geq \delta$, then*

$$\int_{B_{\varepsilon/2}(x)} \left|\frac{\nabla h}{h}\right|^2 \geq A.$$

Proof of 5.4. Since $x \in M_{\text{thick}(\varepsilon)}$ we may assume that $M = \mathbf{H}^3$ and $x = 0$. Suppose that the lemma is false. Then there exists a sequence u_n of positive harmonic functions on $\mathbf{H}^3$ such that $u_n(0) = 1$,

$$\left|\frac{\nabla u_n}{u_n}(0)\right| \geq \delta, \quad \text{and} \quad \int_{B_{\varepsilon/2}(0)} \left|\frac{\nabla u_n}{u_n}\right|^2 \leq \frac{1}{n}.$$

The Harnack inequality [36] assures us that $|\frac{\nabla u_n}{u_n}(x)| \leq R$ for some $R > 0$, all n, and all $x \in \mathbf{H}^3$, so that $u_n(x) \leq e^{Rd(0,x)}$ for all $x \in \mathbf{H}^3$. Therefore, by elliptic theory (see Aubin [2] for example), there exists a subsequence $\{u_j\}$ which converges in the C^1-topology to a positive harmonic function u. But this would imply that

$$\left|\frac{\nabla u}{u}(0)\right| \geq \delta \quad \text{and} \quad \int_{B_{\varepsilon/2}(0)} \left|\frac{\nabla u}{u}\right|^2 = 0,$$

which contradicts the fact that u is C^∞, and completes the proof of Lemma 5.4. q.e.d.

We now recall, from Proposition 3.2, that

$$\int_{C(N)} \left|\frac{\nabla \phi}{\phi}\right|^2 < \infty.$$

Thus $|\frac{\nabla \phi}{\phi}(x)|$ goes to 0 uniformly on $C(N) \cap N_{\text{thick}(\varepsilon)}$, i.e., given $\delta > 0$, there exists a compact submanifold Y_δ of $C(N) \cap N_{\text{thick}(\varepsilon)}$ such that $|\frac{\nabla \phi}{\phi}(x)| \leq \delta$ on $(C(N) \cap N_{\text{thick}(\varepsilon)}) - Y_\delta$. Let M_δ denote $\max\{\phi(x) \mid x \in Y_\delta\}$. Let $L > 0$ be such that if σ is any geodesic in N, then the distance (measured in σ) between components of $\sigma \cap N_{\text{thin}(\varepsilon)}$ is at least L. Let $J > 0$ be a uniform bound on the diameter of each component of $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$.

Lemma 5.5. *If $x \in C(N) \cap N_{\text{thick}(\varepsilon)}$, then*

$$\phi(x) \leq (M_\delta e^{\delta J}) e^{C_1 \delta d(0,x)},$$

where $C_1 = 1 + J/L$.

Proof of 5.5. Let $x \in C(N) \cap N_{\text{thick}(\varepsilon)}$, and let σ be a path joining 0 to x and lying entirely in $C(N) \cap N_{\text{thick}(\varepsilon)}$. We may integrate ϕ, over the portion of σ which does not lie in Y_δ, to obtain

$$\phi(x) \le M_\delta e^{\delta l(\sigma)},$$

where $l(\sigma)$ denotes the length of σ. Now let σ' be the shortest geodesic joining 0 to x; notice that σ' lies entirely in $C(N)$ and $l(\sigma') = d(0, x)$. Now σ' intersects at most $1 + d(0, x)/L$ components of $N_{\text{thin}(\varepsilon)}$. We may replace each component of $\sigma' \cap N_{\text{thin}(\varepsilon)}$ by a path lying entirely in $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$ of length of most J to form a new path $\hat{\sigma}$ joining 0 to x, lying entirely in $C(N) \cap N_{\text{thick}(\varepsilon)}$, and having length at most

$$d(0, x) + \left(1 + \frac{d(0, x)}{L}\right) J = C_1 d(0, x) + J.$$

(Notice that the new path $\hat{\sigma}$ need not be homotopic to the original path.) Therefore,

$$\phi(x) \le M_\delta e^{\delta l(\hat{\sigma})} \le M_\delta e^{C_1 \delta d(0, x) + \delta J},$$

proving Lemma 5.5. q.e.d.

Let $R\colon N \to C(N)$ denote the nearest point retraction, i.e., $R(x)$ is the nearest point of $C(N)$ to x (see Canary-Epstein-Green [7] or Epstein-Marden [17] for a discussion).

Lemma 5.6. *If $R\colon N \to C(N)$ is the nearest point retraction and $0 \in C(N)$, then $d(0, R(x)) \le d(0, x)$ and $\phi(x) \le \phi(R(x))$ for all $x \in N$.*

Proof of 5.6. We may assume that $x \in N - C(N)$. Let P be the totally geodesic hyperplane which passes through $R(x)$ and is perpendicular to the geodesic segment $\overline{xR(x)})$ through x and $R(x)$. Notice that $C(N)$ lies entirely on one side of P (see Epstein-Marden [17]). The geodesic segment $\overline{0R(x)}$ lies entirely within $C(N)$ and thus makes an obtuse angle with the geodesic segment $\overline{xR(x)}$. Therefore by considering the geodesic triangle with vertices 0, x, and $R(x)$ we see that $d(0, R(x)) < d(0, x)$.

We will now see that $|\alpha'_x(\xi)| \le |\alpha'_{R(x)}(\xi)|$ for all $\xi \in L_\Gamma$, which clearly implies that $\phi(R(x)) \ge \phi(x)$. If $\xi \in L_\Gamma$, then the geodesic half ray $\overline{R(x)\xi}$ lies entirely in $C(N)$ and is perpendicular to the horoball

$$H = \left\{ y \in \mathbf{H}^3 \,\middle|\, \frac{1 - |y|^2}{|y - \xi|^2} = \frac{1 - |R(x)|^2}{|R(x) - \xi|^2} \right\}$$

based at ξ and passing through $R(x)$. Therefore, since $\overline{R(x)\xi}$ makes an obtuse angle with $\overline{xR(x)}$, we have

$$|\alpha'_x(\xi)| = \frac{1-|x|^2}{|x-\xi|^2} \le \frac{1-|R(x)|^2}{|R(x)-\xi|^2} = |\alpha'_{R(x)}(\xi)|,$$

which completes the proof of Lemma 5.6. q.e.d.

We now deal with compact components of $N_{\text{thin}(\varepsilon)}$.

Lemma 5.7. *If T is any compact component of $N_{\text{thin}(\varepsilon)}$ and $x \in T$, then*

$$\phi(x) \le (M_\delta e^{\delta(C_1 J + J)}) e^{C_1 \delta d(0,x)}.$$

Proof of 5.7. Let S be the boundary of T. The maximum principle (cf. Aubin [2]) implies that the maximum of ϕ over T occurs at a point $\hat{x}$ on S. Lemma 5.6 shows that $\hat{x} \in S \cap C(N)$ (since if $y \in T$, then $R(y) \in T$). Consider $\hat{\sigma}$, the shortest geodesic joining 0 to x. Notice that $\hat{\sigma}$ lies entirely within $C(N)$, and let y be the first point of intersection of $\hat{\sigma}$ with T. Since y is within J of $\hat{x}$, we have $d(0,\hat{x}) \le d(0,x) + J$. We may then apply Lemma 5.5 to see that

$$\phi(x) \le \phi(\hat{x}) \le (M_\delta e^{\delta J}) e^{C_1 \delta d(0,x) + C_1 J \delta} = (M_\delta e^{\delta(C_1 J + J)}) e^{C_1 \delta d(0,x)},$$

proving Lemma 5.7. q.e.d.

We now need only deal with noncompact portions of $N_{\text{thin}(\varepsilon)}$. In §2 of [32] Sullivan establishes that the eigenfunction corresponding to Patterson-Sullivan measure behaves roughly like $e^{(2-\delta)d(0,x)}$ on torus cusps and like $e^{(1-\delta)d(0,x)}$ on rank-one cusps of bounded type (see also Patterson [27]). To both be precise and avoid introducing the construction of the Patterson-Sullivan measure we will use an explicit version of Sullivan's result, which is obtained in the proof of Theorem 3.5.9 in Nicholls [26].

If M is any complete hyperbolic 3-manifold and T is any noncompact component of $M_{\text{thin}(\varepsilon)}$ having boundary S, then there exists a map $F_T \colon T \to S$ which takes a point $x \in T$ to the nearest point on S. (If Γ_∞ is a group of parabolic elements preserving ∞ in the upper half-space model for $\mathbf{H}^3$ and T is isometric to $\{(z,t) \mid t \ge 1\}/\Gamma_\infty$, then $F(z,t) = (z,1)$.)

Lemma 5.8. *Let M be a complete hyperbolic 3-manifold, ν its associated Patterson-Sullivan measure, and T a bounded cusp of rank k. Then given any point y in the boundary S of T and any $\alpha > 0$ there exists D such that if $p \in F_T^{-1}(y)$ then*

$$\phi_{\nu,\delta}(p) \le D e^{(k+\alpha-\delta)d(p,y)},$$

where δ is the critical exponent of the Poincaré series.

We then improve this slightly to obtain:

Lemma 5.9. *If T is any noncompact component of N, then given any $\alpha > 0$ there exists $B(T, \alpha)$ such that if $x \in T \cap C(N)$, then*

$$\phi(x) \leq B(T, \alpha)e^{\alpha d(0,x)}.$$

Proof of 5.9. Pick $y \in S \cap C(N)$, and let D be such that if $p \in F_T^{-1}(y)$, then $\phi(p) \leq De^{\alpha d(y,p)}$. If $x \in T \cap C(N)$, then there exists a point $\hat{x} \in F_T^{-1}(y)$ such that $d(\hat{x}, x) \leq J$. Now notice that $d(\hat{x}, y) \leq d(0, x) + 2J$, therefore $\phi(\hat{x}) \leq De^{\alpha(d(0,x)+2J)}$. But, since $|\frac{\nabla\phi}{\phi}(y)| \leq R$ for all $y \in N$ and $d(\hat{x}, x) \leq J$,

$$\phi(x) \leq e^{RJ}\phi(\hat{x}) \leq De^{RJ+2\alpha J}e^{\alpha d(0,x)},$$

from which we obtain the assertion in Lemma 5.9. q.e.d.

(Notice that this is the only point at which we have used the construction of Patterson-Sullivan measure; if N has no cusps, then the proof applies when μ is any measure supported on the limit set satisfying condition $(*)$.)

Let $B(C_1\delta)$ denote the maximum of $B(T, C_1\delta)$ taken over the (finitely many) noncompact components of $N_{\text{thin}(\varepsilon)}$. Recall from Lemma 5.6 that $\phi(x) \leq \phi(R(x))$, and from Lemma 5.2 that $d(0, x) \leq d(0, R(x))$. Thus, by combining Lemmas 5.5, 5.7, and 5.9 we see that

$$\phi(x) \leq \phi(R(x)) \leq (M_\delta e^{\delta(C_1 J+J)} + B(C_1\delta))e^{C_1\delta d(0,x)}$$

for all points $x \in N$. Therefore, $e(\phi) \leq C_1\delta$, but since this is true for all $\delta > 0$, $e(\phi) \leq 0$. This completes the proof of Proposition 5.3 and hence of Theorem C.

Remarks. (1) Let N be a hyperbolic 3-manifold homeomorphic to $S \times \mathbf{R}$ whose convex core is homeomorphic to $S \times [0, \infty)$ and which has a uniform lower bound on its injectivity radius. N is said to be a "hyperbolic half-cylinder" if there exists an embedded surface $\widehat{S}$, homotopic to $S \times \{0\}$, such that if $\rho(x)$ denotes the distance from x to $\widehat{S}$, there exists K such that given any n there exists $d \in [n, n+1]$ such that the portion of $\rho^{-1}(d)$ contained in the convex core has diameter less than K. With these assumptions, Sullivan proves that ϕ has linear growth on the convex core, and that Patterson-Sullivan measure is ergodic, hence unique.

(2) Examples of topologically tame hyperbolic 3-manifolds which are not geometrically finite but do have a lower bound on their injectivity radius may be given by using the techniques of Thurston [35] or Jorgenson. The space $QF(S)$ of geometrically finite hyperbolic structures without cusps on $S \times R$ is parametrized by $\mathscr{T}(S) \times \mathscr{T}(S)$, where $\mathscr{T}(S)$ denotes the space of marked hyperbolic structures on the closed surface

S of genus $g \geq 2$. If (σ, τ) is any point in $QF(S)$, and ϕ is any pseudo-Anosov homeomorphism of S, then the sequence of hyperbolic manifolds $(\sigma, \phi^n(\tau))$ converges, both geometrically and algebraically (at least up to subsequence), to a topologically tame hyperbolic 3-manifold with a lower bound on its injectivity radius. These examples have limit sets of measure zero. (Recall that a homeomorphism $\phi: S \to S$ is said to be pseudo-Anosov if it is not homotopic to a finite order homeomorphism and no finite collection of disjoint simple closed curves on S is preserved up to isotopy by ϕ.) This construction provides a $(6g-6)$-dimensional space of hyperbolic 3-manifolds with a lower bound on there injectivity radius, however one still expects such examples to be rare in the boundary of $QF(S)$.

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References

[1] L. V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964) 413–29.

[2] T. Aubin, *Nonlinear analysis of manifolds. Monge-Ampère equations*, Springer, New York, 1982.

[3] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) **124** (1986) 71–158.

[4] P. Braam, *A Kaluza-Klein approach to hyperbolic three-manifolds*, Enseignement Math. **34** (1988) 275–311.

[5] M. Burger & R. Canary, in preparation.

[6] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École. Norm. Sup. **15** (1982) 213–230.

[7] R. D. Canary, D. B. A. Epstein & P. Green, *Notes on notes of Thurston*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 3–92.

[8] R. D. Canary, *Ends of hyperbolic* 3-*manifolds*, preprint.

[9] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis, Princeton University Press, Princeton, NJ, 1970, 195–199.

[10] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975) 289–297.

[11] S. Y. Cheng & S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975) 333–354.

[12] M. Culler & P. Shalen, *Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic* 3-*manifolds*, J. Amer. Math. Soc. **5** (1992) 231–288.

[13] J. Dodziuk & B. Randol, *Lower bounds for* λ_1 *on a finite-volume hyperbolic manifold*, J. Differential Geometry **24** (1986) 133–139.

[14] H. Donnelly, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981) 1–14.

[15] P. Doyle, *On the bass note of a Schottky group*, Acta. Math. **160** (1988) 249–284.

[16] C. L. Epstein, *The spectral theory of geometrically periodic hyperbolic 3-manifolds*, Mem. Amer. Math. Soc. No. 335 (1985).

[17] D. B. A. Epstein & A. Marden, *Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 113–253.

[18] M. Feighn & D. McCullough, *Finiteness conditions for 3-manifolds with boundary*, Amer. J. Math. **109** (1987) 1155–69.

[19] T. Jorgensen, A. Marden & B. Maskit, *The boundary of classical Schottky space*, Duke Math. J. **46** (1979) 441-6.

[20] P. Lax & R. S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*, J. Funct. Anal. **46** (1982) 280–350.

[21] ——, *Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces*. I, II, III, Comm. Pure Appl. Math. **37** (1984) 303–324, 779–813; **38** (1985) 197–208.

[22] P. Li & S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201.

[23] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) **99** (1974) 383–462.

[24] B. Maskit, *Kleinian groups*, Springer, Berlin, 1988.

[25] J. W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, The Smith Conjecture (J. Morgan and H. Bass, eds.), Academic Press, New York, 1984, 37–125.

[26] P. J. Nicholls, *The ergodic theory of discrete groups*, Cambridge University Press, Cambridge, 1989.

[27] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976) 241–273.

[28] ——, *Lectures on limit sets of Kleinian groups*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 281–323.

[29] R. Schoen, *A lower bound for the first eigenvalue of a negatively curved manifold*, J. Differential Geometry **17** (1982) 233–238.

[30] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Haute Études Sci. Publ. Math. **50** (1979) 419–450.

[31] ——, *Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension* 2, Geometry Symposium (Utrecht 1980), Lecture Notes in Math., Vol. 894, Springer, Berlin, 1981, 127–144.

[32] ——, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta. Math. **153** (1984) 259–277.

[33] ——, *Aspects of positivity in Riemannian geometry*, J. Differential Geometry **25** (1987) 327–351.

[34] W. Thurston, *The geometry and topology of* 3*-manifolds*, lecture notes.

[35] ——, *Hyperbolic structures on* 3*-manifolds*, II: *Surface groups and* 3*-manifolds which fiber over the circle*, preprint.

[36] S. T. Yau, *Harmonic function on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975) 201–228.

[37] ——, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976) 659–670.

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